Shape Modelling for Computer Graphics

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Unit materials

- Lecture notes
- Seminar handouts

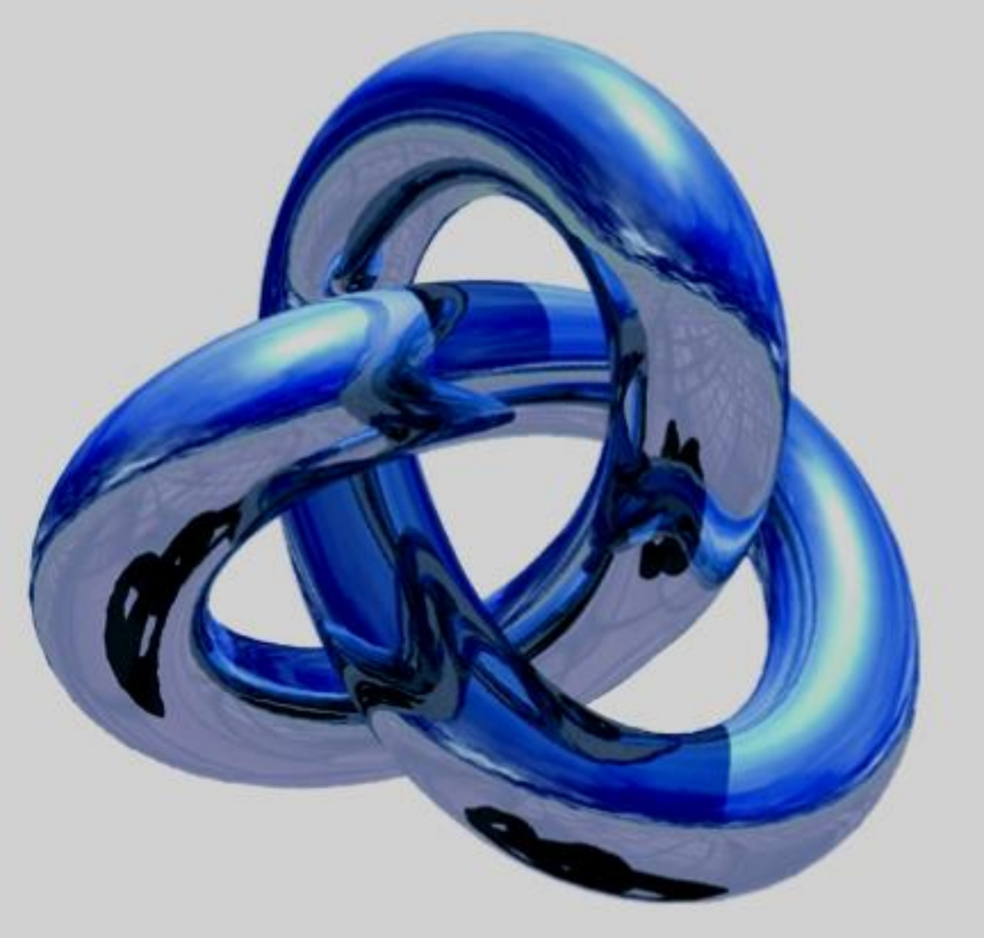
are available at

http://gm.softalliance.net/

Advice: download and print lecture notes before the next lecture

Introduction to Shape Modelling

basic notions of linear algebra, analytical geometry and set theory



Contents

- "Shape" term definitions
- Shape and point notions
- Vector and affine spaces
- Coordinates and metrics
- Shape dimension
- Defining a point set
- Reference materials

Modelling Notions

- Shape and point notions
- Spaces:

✓ linear space

✓ affine space

✓ metric space

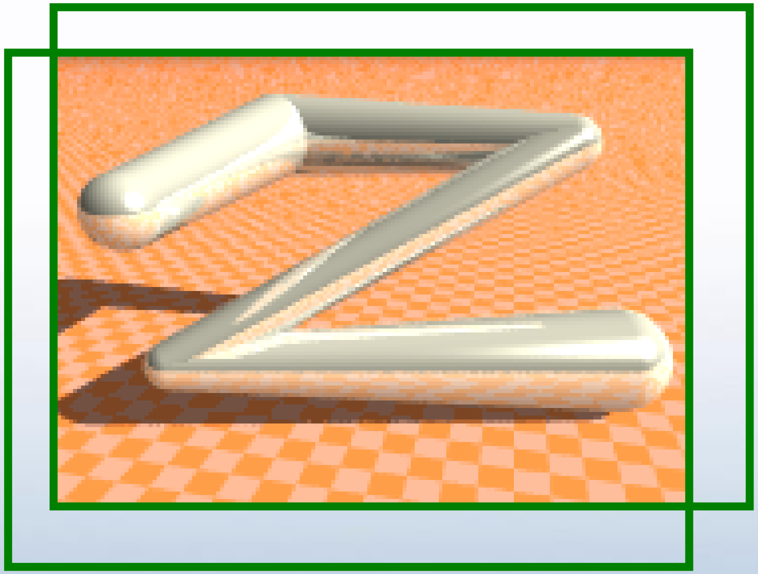
✓ Euclidean space

space dimensions

✓ vector space

dot product and cross product

- Affine and Cartesian coordinates
- Descartes' conception of geometry
- Point set



Shape Definitions

- Shape means the outer form of something, that you see or feel (Longman Dictionary of Contemporary English).
- Geometry is built "instead of points, straight lines, and planes – tables, chairs, and beer mugs " (D. Hilbert)
- Shape is the spatial arrangement of something as distinct from its substance (Princeton Word Net)

David Hilbert

(1862-1943)

Shape Definitions

Felix Christian Klein

(1849-1925)

- A geometrical figure (shape) is a set of points.
- Shape is the geometric information invariant to a particular class of transformations (F. Klein).

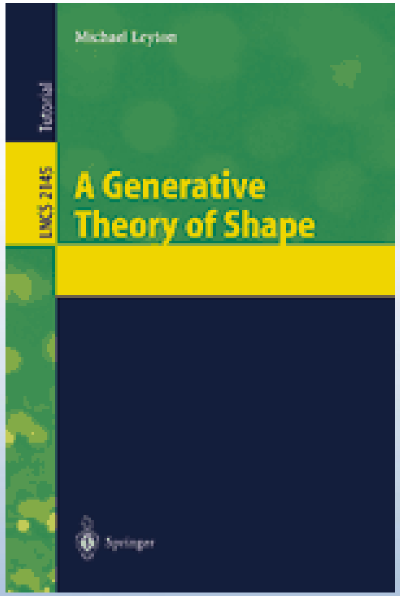
In “Erlangen program” in 1872, Klein proposed a new unified approach to geometry by starting with a set, and a group of transformations of the set. The properties of the geometry are those properties which are invariant under the group.

Shape Definitions

- The generative theory of shape represents a given shape by a program that generates a point set. The program must be inferable (recoverable) from the point set (M. Leyton).

Shape acts as a memory store for the generative operations of the program:

Geometry ≡ Memory storage



A Generative Theory of Shape, 2001

EU AIM@SHAPE Project

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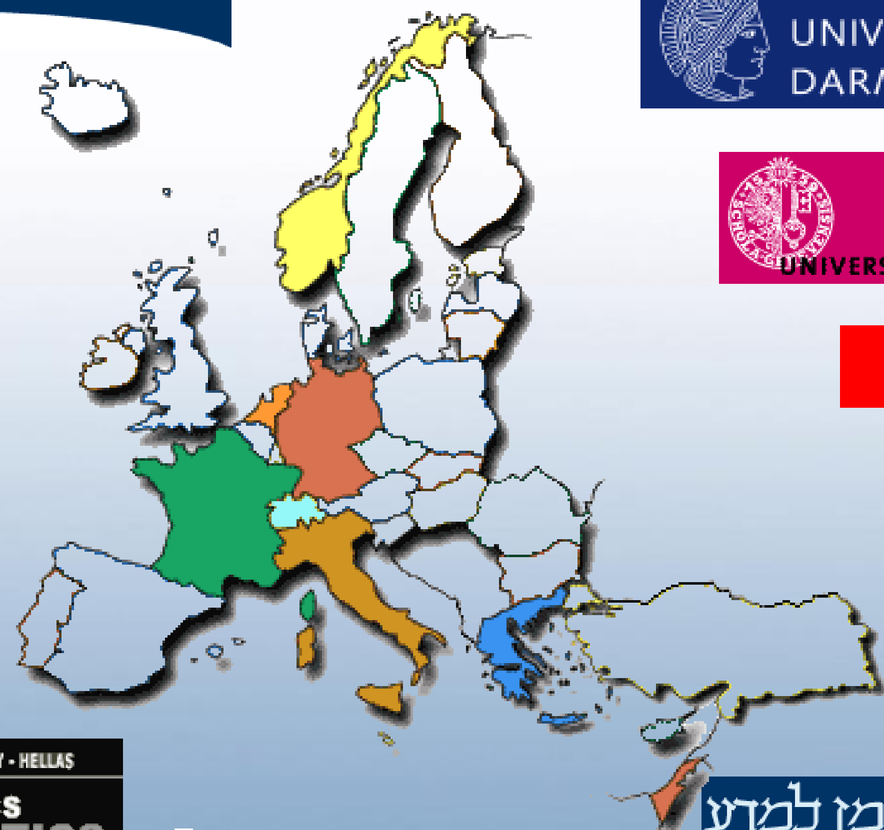


Project AIM@SHAPE

Shape = any individual object having a visual appearance which exists in some (two-, three- or higher- dimensional) space (pictures, sketches, images, 3D objects, videos, 4D animations, ...)

Shapes

- ✓ have a **geometry** (the spatial extent of the object)
- ✓ can be described by **structures** (object features and part-whole decomposition)
- ✓ have **attributes** (colours, textures, names, attached to an object, its parts and/or its features)
- ✓ have a **semantics** (meaning, purpose)
- ✓ may also have **interaction with time** (history, shape morphing, animation, video)

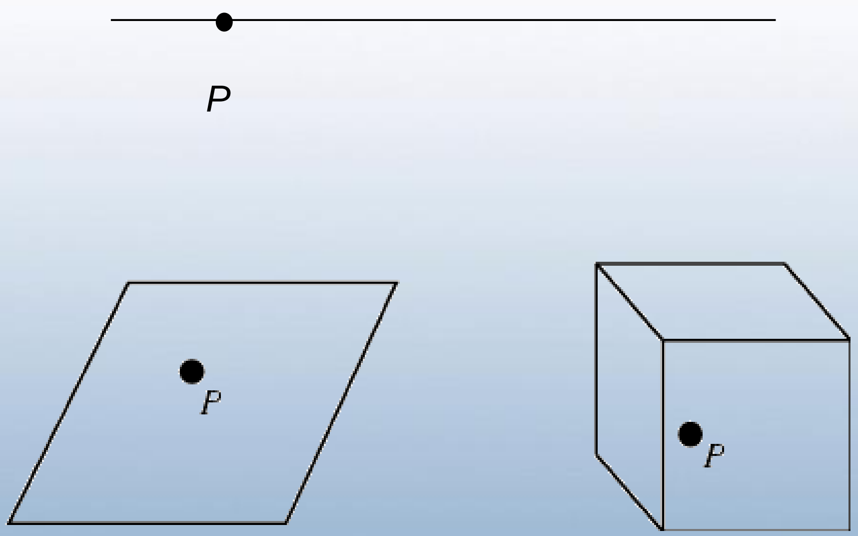
Shape and Point Notions

We will consider a shape as a point set in n-dimensional geometric space, for example, in Euclidean space.

Shape and Point Notions

Two approaches to point definition:

- Geometric (Euclid)
- Algebraic (linear algebra and analytical geometry)



Geometric Point Notion

Euclid "Elements Book 1. Definitions, Postulates and Common Notions":

Definition 1. A point is that which has no part.

Euclid treats a point as having no width, length, or breadth, but as an indivisible location.

Euclid
(325BC-265BC)

A point is undefined fundamental term of geometry

Algebraic Point Notion

A point is considered an element of an affine space.

An affine space is a set with every ordered pair of points associated with an element of some linear space (vector space).

A linear space is a set closed under operations of element addition and scalar multiplication, satisfying certain conditions.

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Vector space

- Two types of elements:
  - Scalars (real numbers): α, β, …
  - Vectors (n-tuples): u, v, w, …
- Operations on vectors:
  - Multiplication by scalar
  - Addition
  - Subtraction

Scalars

- Scalars: $\alpha, \beta, \ldots$
- Addition and multiplication (+ and $\bullet$) operations defined
- Scalar operations are
  - Associative: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
  - Commutative: $\alpha + \beta = \beta + \alpha$

    $\alpha \bullet \beta = \beta \bullet \alpha$
  - Distributive: $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma$

    $\alpha \bullet (\beta + \gamma) = (\alpha \bullet \beta) + (\alpha \bullet \gamma)$

- Additive Identity = 0

$$\alpha + 0 = 0 + \alpha = \alpha$$

- Multiplicative Identity = 1

$$\alpha \bullet 1 = 1 \bullet \alpha = \alpha$$

- Additive Inverse = $-\alpha$

$$\alpha + (-\alpha) = 0$$

- Multiplicative Inverse= α^{-1}

$$\alpha \bullet \alpha^{-1} = 1$$

Vector Operations

Vector addition:

$$u + v = w$$

✓ **Commutative**

$$u + v = v + u$$

✓ **Associative**

$$(u + v) + w = u + (v + w)$$

✓ **Additive identity**

There is a vector 0, such that for all u,

$$0 + u = u = u + 0$$

✓ **Inverse**

For any u there is a vector -u such that

$$u + (-u) = 0$$

✓ **Scalar multiplication**

$$\alpha u, \alpha = const$$

In order for V to be a vector space, the following conditions must hold for all elements u, v, w $\in$ V and any scalars r, s :

1. Commutativity: $u + v = v + u$
2. Associativity of vector addition: $(u + v) + w = u + (v + w)$
3. Additive identity: For all u, $0 + u = u = u + 0$
4. Existence of additive inverse: For any u, there exists a -u such that
   $u + (-u) = 0$
5. Associativity of scalar multiplication: $r (s u) = (r s) u$
6. Distributivity of scalar sums: $(r + s) u = r u + s u$
7. Distributivity of vector sums: $r (u + v) = r u + r v$
8. Scalar multiplication identity: $1 \cdot u = u$

Basis vectors and coordinates of vector

Two vectors are linearly dependent when one is a multiple of the other.

A basis of an n-dimensional vector space is defined as a set of n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that are linearly independent.

Every vector **v** in space can be written as a linear combination of the basis vectors

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n,$$

where $a_1, a_2, \ldots, a_n$ are called coordinates of the vector **v**.

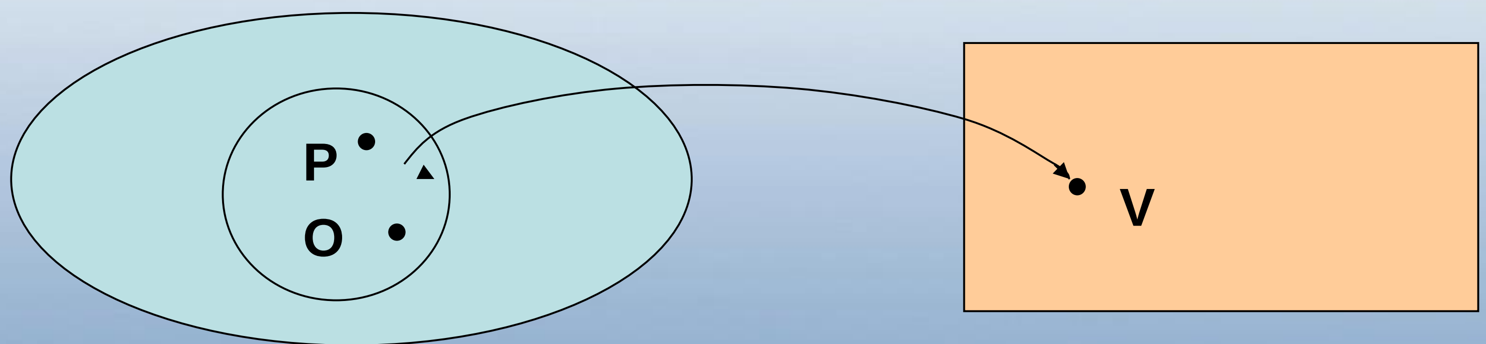
Space Dimension

Dimension of affine space is equal to dimension of associated linear space.

Dimension of linear space is equal to maximal number of linearly independent vectors.

Affine Coordinates

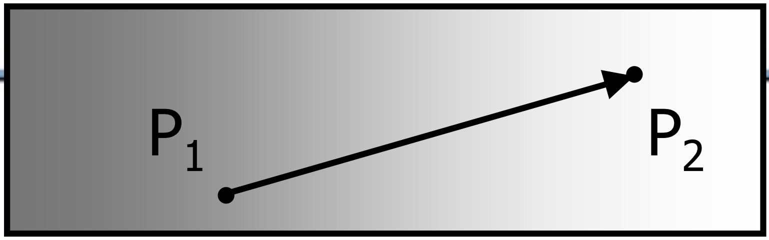
In the associated affine space we can select one point O as an origin. Then for any point P, coordinates of the vector **v** associated with the pair (O,P) are called affine coordinates.



Affine space

Vector space

Vector



- An element of a vector (linear) space
- A quantity in which both the magnitude and the direction must be stated
- Can be represented as a directed straight line segment
- Consider all locations in relationship to one central reference point, called origin.
- Vector has direction according to origin and length in nD space
- Can be defined by

1) Two linear scalar arrays as start and end points or
2) One linear scalar array (end point) with (0,0,...,0) as start point by default

Vectors are used extensively in computer graphics to:

– represent positions of vertices of objects

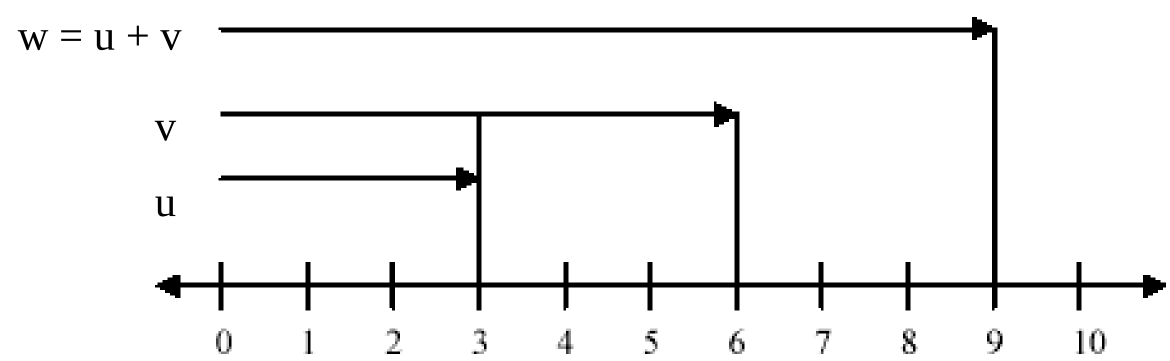
– determine orientation of a surface in space ("surface normal")

– represent relative distances and orientations of lights, objects, and viewers in a 3D scene (vectors from light sources to surfaces)

- represent force, velocity, flow, etc.

Vector operations illustrated

Vector addition in R^1



$u = [3], \; v = [6], \; w = [9]$

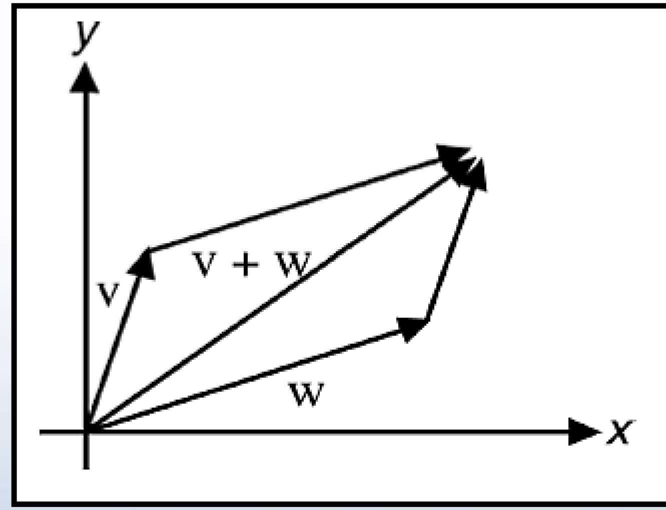
Like as addition of real numbers

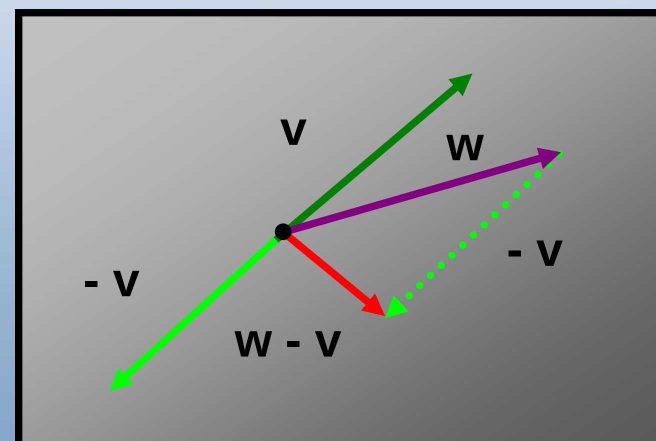
Vector operations illustrated

Vector w added
(or subtracted) to vector u,
using the parallelogram rule:

- drawing the vector *v*, then
- drawing the vector *w*, taking care to place the tail of vector *w* at the head of vector *v*, and finallv
- drawing a vector *v* + *w* from the free tail of vector *v* to the free head of vector *w*

$$v + w = w + v$$



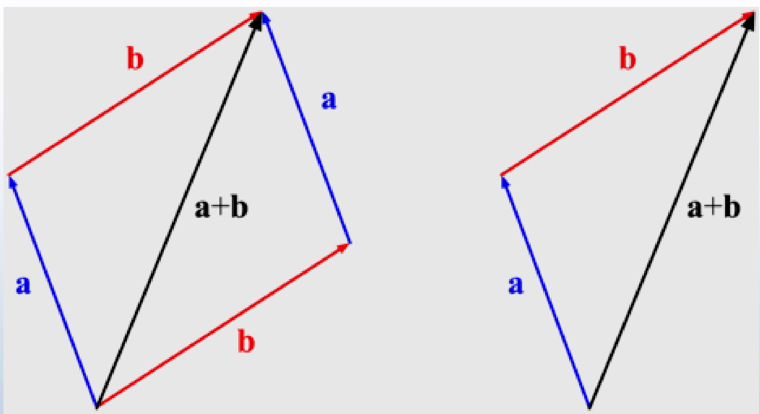


Vector operations illustrated

Let **a**=*a*1**i** + *a*2**j** + *a*3**k** and **b**=*b*1**i** + *b*2**j** + *b*3**k**

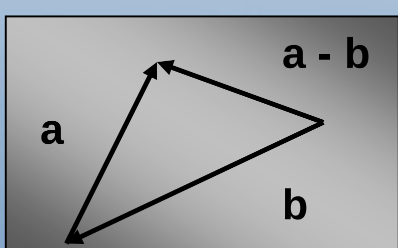
The **sum** of **a** and **b** is:

http://en.wikipedia.org



$$\mathbf{a}+\mathbf{b}=(a_1+b_1)\mathbf{i}+(a_2+b_2)\mathbf{j}+(a_3+b_3)\mathbf{k}$$

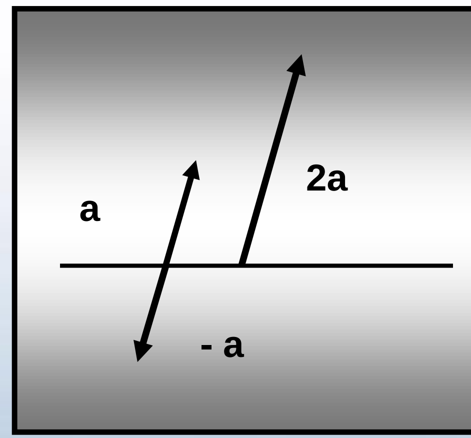
$$\mathbf{a}-\mathbf{b}=(a_1-b_1)\mathbf{i}+(a_2-b_2)\mathbf{j}+(a_3-b_3)\mathbf{k}$$



Subtraction

a - **b** = **a** + (-1)**b**

Scalar multiplication



$$r\mathbf{a}=(ra_1)\mathbf{i}+(ra_2)\mathbf{j}+(ra_3)\mathbf{k}$$

The length of *r***a** is |*r*||**a**|

Geometries

- Affine Geometry
  - Scalars + Points + Vectors and their operations

- Euclidian Geometry
  - Affine Geometry lacks angles, distance
  - New operations: Inner/ dot product, which gives length, distance, normalization, angle, orthogonality.

Dot Product

Dot product of two vectors is a real value,

not a vector!

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Example:

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = ax + by + cz + dw$$

Dot Product

- The dot product of two vectors u and v (inner product or, since its result is a scalar, the scalar product) is also defined as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \| \boldsymbol{u} \| \, \| \boldsymbol{v} \| \, \boldsymbol{Cos}\, \theta$$

- where ||**u**|| and ||**v**|| denote the norm (or length) of **u** and **v**, and θ is the measure of the angle between **u** and **v.**
- Geometrically, this means that **u** and **v** are drawn with a common start point and then the length of **u** is multiplied with the length of that component of **v** that points in the same direction as **u**.

Uses of the dot product

- Define length or magnitude of a vector
- Normalize vectors (generate vectors whose length is 1, called unit vectors)
- Measure angles between vectors
- Determine if two vectors are perpendicular
- Find the length of a vector projected onto a coordinate axis

Length of a Vector

The dot product of a vector with itself, $(v \bullet v)$, is the square of the length of the vector:

We define the norm of a vector (i.e., its length) to be
$\|v\| = \sqrt{v \bullet v}$

Thus, $(v \bullet v) \geq 0$ for all v, with $(v \bullet v) = 0$ if and only if $v = 0$

v called a unit vector if $\|v\| = \sqrt{v \bullet v}$ **= 1** is denoted $\hat{v}$

To make an arbitrary vector v into a unit vector, i.e. to "normalize" it, divide by the length (norm) of v, which is denoted ||v||. Note that if v = 0, then its unit vector is undefined. So in general (with the 0 exception) we have:

$$\hat{v} = \frac{1}{\|v\|} v$$

Standard basis vectors

- The unit vectors (i.e., whose length is one) on the x and y-axes are called the standard basis vectors of the plane
- The collection of all scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the first coordinate axis
- The collection of all scalar multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the second coordinate axis
- Then any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ can be expressed as the sum of scalar multiples of the unit vectors:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We call these two vectors basis vectors because any other vector can be expressed in terms of them

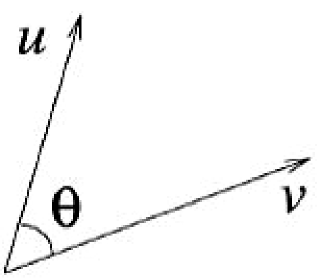
Projection and angle between two vectors

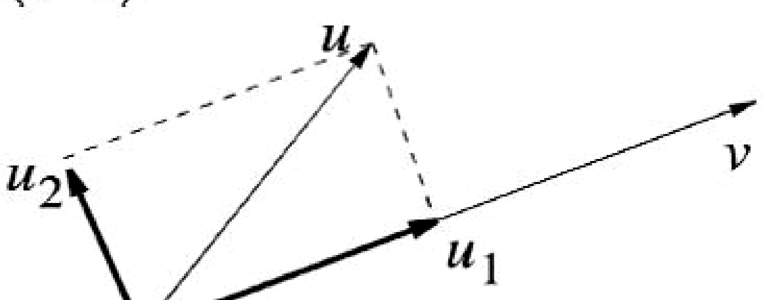
- *Angle between vectors,* θ

$$\theta = \mathrm{ang}(\vec{u},\vec{v}) = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}(\hat{u}\cdot\hat{v}).$$

- *Projection of vectors*

$$\vec{u}_1 = \frac{(\vec{u}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v} \qquad \vec{u}_2 = \vec{u} - \vec{u}_1.$$





Pics/Math courtesy of Dave Mount @ UMD-CP

Orthogonality

Scalar product of two vectors x and y:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots . + x_n y_n$$

Two vectors are orthogonal (perpendicular) if

$$x \cdot y = 0$$

In 3D space, three vectors can be mutually orthogonal and linearly independent.

Two subspaces *A* and *B* of vector space are called orthogonal subspaces if each vector in *A* is orthogonal to each vector in B.

Cross Product

The cross product (also vector product or outer product) of two vectors is a vector.

The cross product is only meaningful in three dimensions.

The cross product $u \times v$ is a vector perpendicular to both u and v and is defined as:

$$u \times v = ||u||\ ||v|| \sin\theta\ \mathbf{n}$$

where θ is the measure of the angle between u and v, and $\mathbf{n}$ is a unit vector perpendicular to both u and v.

Cross product

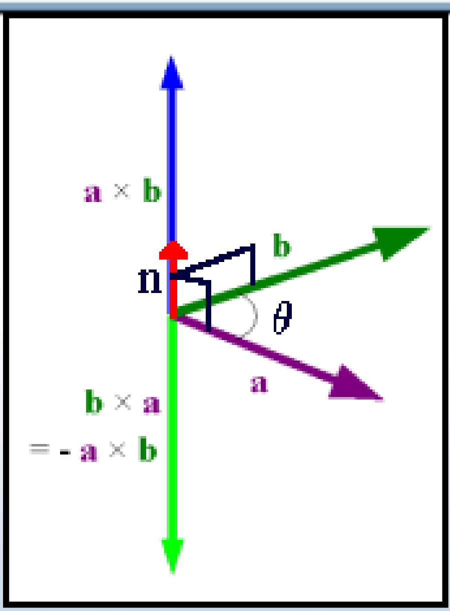
- Given two non-parallel vectors, A and B
- A x B calculates third vector C that is orthogonal to A and B

$$A \times B = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$A \times B = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)$$

Cross Product

- The problem is that there are two unit vectors perpendicular to both **b** and **a**. Which vector is the correct one depends upon the *orientation* of the vector space, on the handedness of the vector triple.
- The vector triple i, j, k is called right handed, if the three vectors are situated like the thumb, index finger and middle finger (pointing straight up from your palm) of your right hand.



Metric space

A non-negative function g(x,y) describing the "distance" between two points for a given set is called a metric.

It satisfies the triangle inequality:

$$g(x,y) + g(y,z) \geq g(x,z)$$

and is symmetric: $g(x,y) = g(y,x)$

also satisfies: $g(x,x) = 0$

An affine space with a metric is called a metric space.

Euclidean space

The Euclidean metric is the function $d(x,y)$ that assigns to any two points $x = (x_1,\ldots, x_n)$ and $y = (y_1,\ldots, y_n)$ of affine space the number

$$d(x,y)=\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$

and gives the "standard" distance between any two points.

Example: Euclidean space E^n is the space of all n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$ with the Euclidean metric.

Euclidean distance

Euclidian Distance from (x,y) to $(0,0)$

$\sqrt{x^2 + y^2}$ in general: $\sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}$

which is just: $\sqrt{\vec{x} \bullet \vec{x}}$

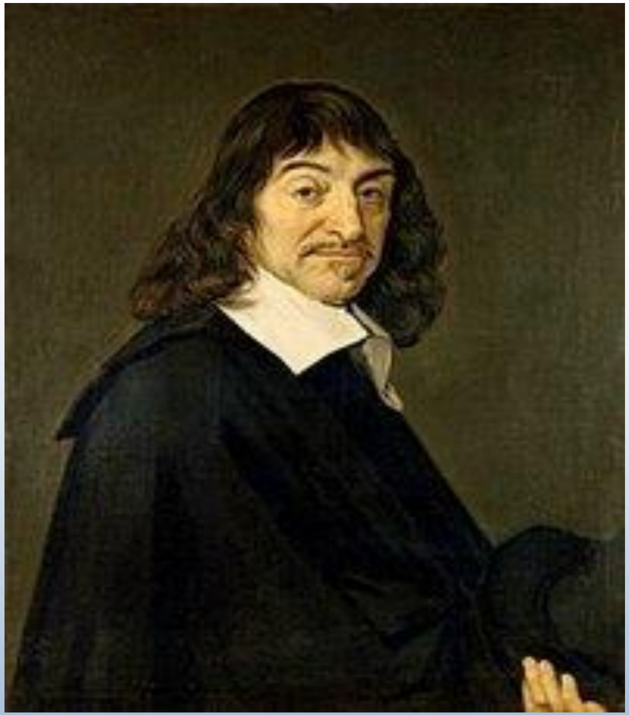
This is also the length of vector $\underline{v}$:

$||\underline{v}||$ or $|\underline{v}|$

Normalization of a vector: $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$.

Orthogonal vectors: $\vec{u} \cdot \vec{v} = 0$

Descartes' conception of geometry



René Descartes

(1596-1650)

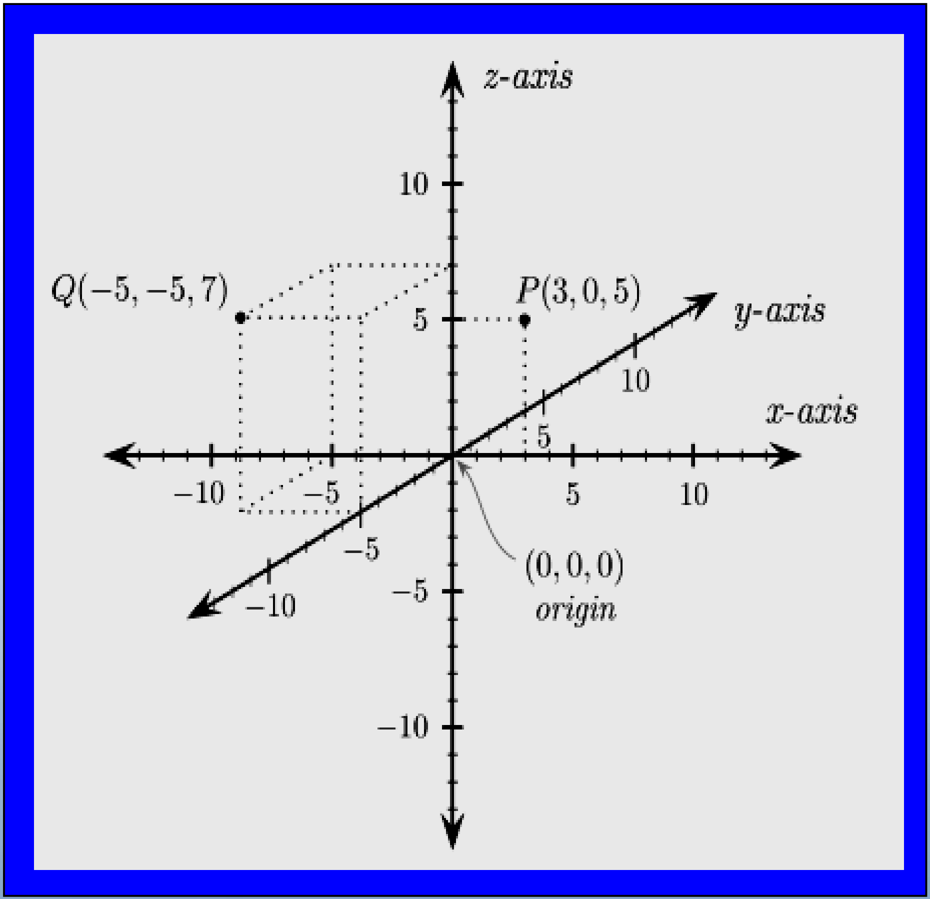
The term "Cartesian" is used to refer to René Descartes' conception of geometry (“*Discourse on Method*” and “*La Géométrie*” 1637), which is based on the representation of points in the plane by ordered pairs of real numbers - Cartesian coordinates.

This idea allows geometric relations in analytical geometry to be expressed by means of algebraic equalities.

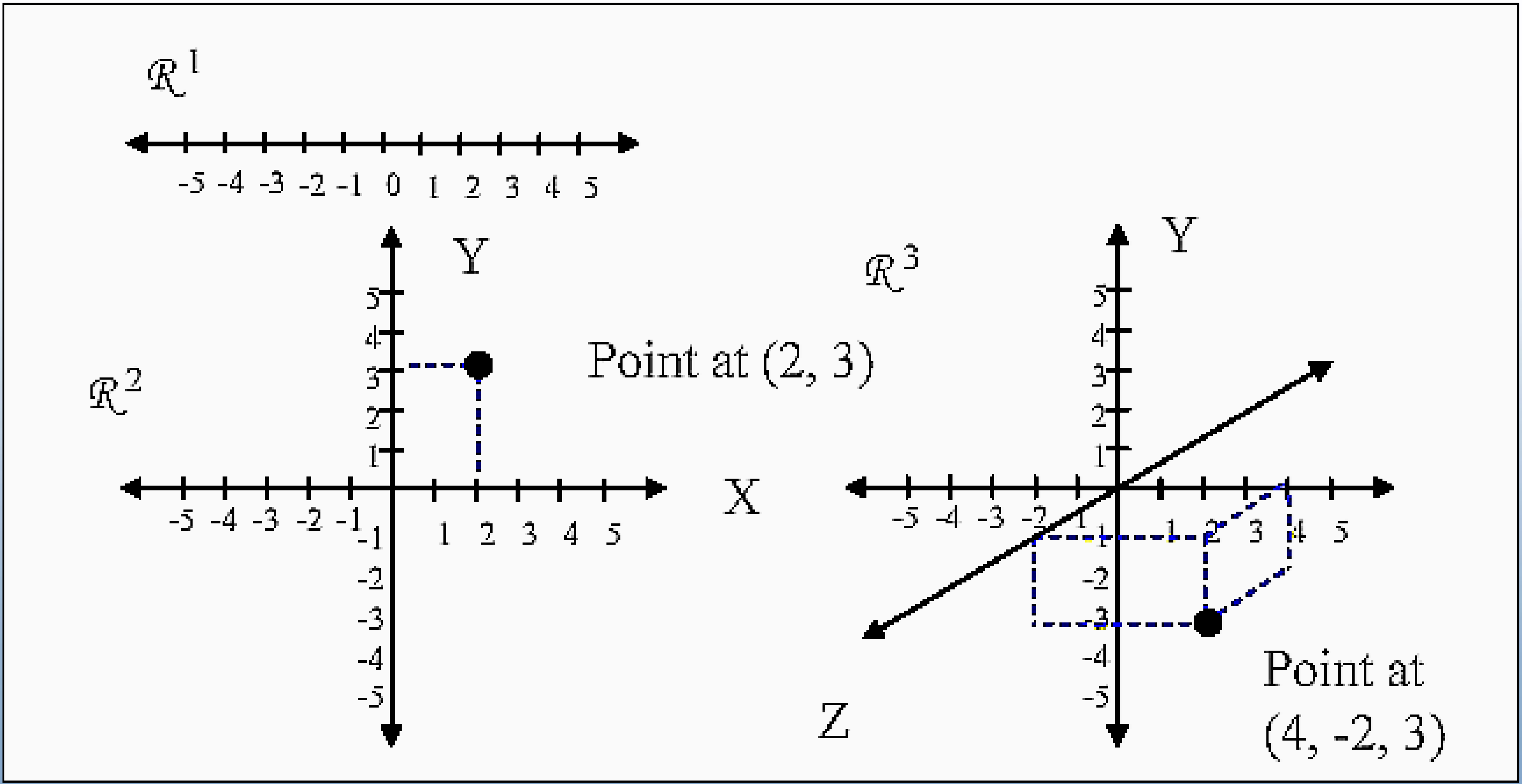
Cartesian coordinates

Cartesian coordinates in 2D or 3D space are defined by two or three mutually orthogonal lines (axes) intersecting in the origin.

Point coordinates are taken as distances from the origin along each axes.



One, two and three dimensional real coordinate systems



Thank you for your attention!