Discrete Mathematics

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Unit materials

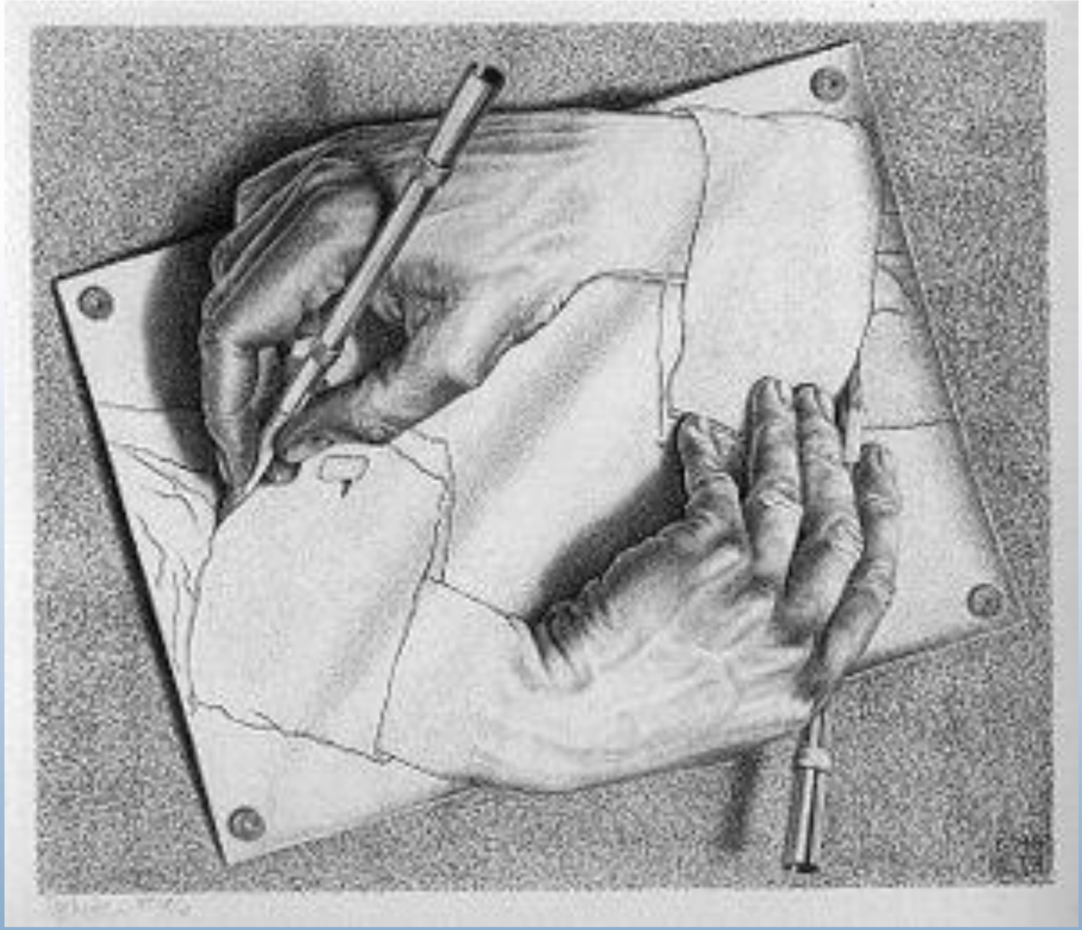
- Lecture notes
- Seminar handouts

are available at

http://gm.softalliance.net/

Advice: download and print lecture notes before the next lecture

Functions



Drawing Hands is a lithograph by the Dutch artist M. C. Escher, printed in January 1948.

Functions

- From calculus, you know the concept of a real-valued function *f*,
  which assigns to each number $x \in \mathbf{R}$
  one particular value $y = f(x)$, where $y \in \mathbf{R}$.
  - *Example*: *f* defined by the expression

$$f(x)=x^2$$

- The notion of a function can be generalized to the concept of assigning elements of *any* set to elements of *any* set.

Function: Formal Definition

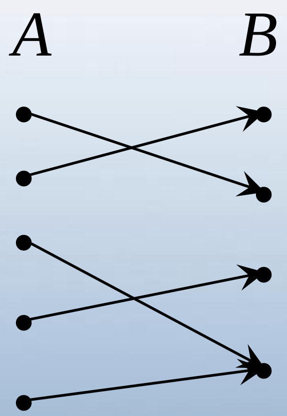
- For any sets *A*, *B*, we say that a *function f (or "mapping") from A to B* ($f{:}A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Some further generalizations of this idea:
  – A *partial* (non-*total*) function *f* assigns *zero or one* elements of *B* to each element $x \in A$.
  – Functions of *n* arguments; relations.

Basic Properties of Functions

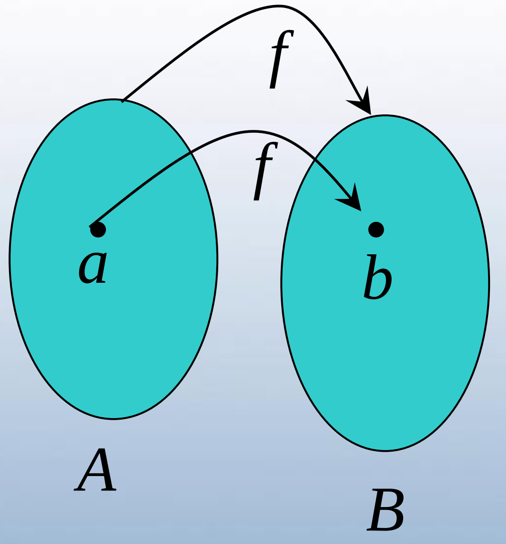
- We can represent a function f:$A \rightarrow B$ as a set of ordered pairs $f = \{(a,f(a)) \mid a \in A\}$.
- This makes f a relation between A and B: f is a subset of A x B. But functions are special:
  - for every $a \in A$, there is at least one pair (a,b). Formally:
    $\forall a \in A \exists b \in B((a,b) \in f)$
  - for every $a \in A$, there is at most one pair (a,b). Formally:
    $\neg \exists a,b,c((a,b) \in f \wedge (a,c) \in f \wedge b \neq c)$

Graphs of Functions

- Functions can be represented graphically in several ways:



Bipartite Graph

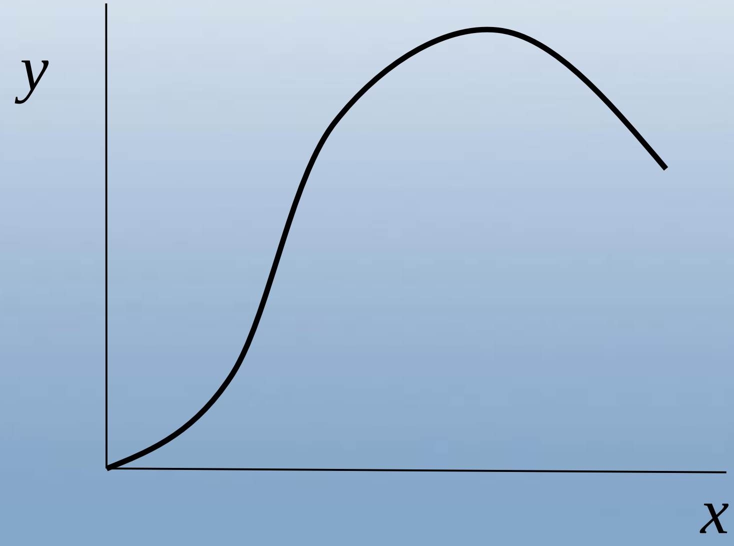


Like Venn diagrams

Graphs of Functions

- A relation over numbers can be represented as a set of points on a plane. (A point is a pair (x,y).)
- A function is then a curve (set of points), with only one y for each x.

Plot



Some Function Terminology

- If f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then we say:
  - – A is the *domain* of f.
  - – B is the *codomain* of f.
  - – b is the *image* of a under f.
  - – a is a *pre-image* of b under f.
    - In general, b may have more than one pre-image.
  - – The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\ f(a)=b\}$.

Range versus Codomain

- The range of a function may not be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: “*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}.”
- At this point, you know *f*’s codomain is: {A,B,C,D,E}__________, and its range is unknown_________.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}__________, but its codomain is still {A,B,C,D,E}_________________.

(n-ary) Functions on a Set

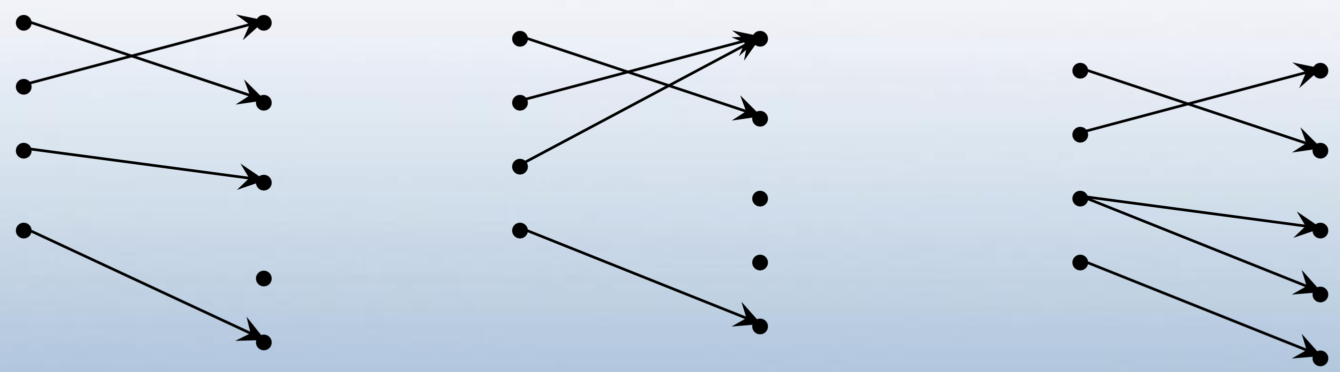
- An *n*-ary function (also: n-ary *operator) over S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- Examples:
- if *S*={**T**,**F**}, $\neg$ can be seen as a unary operator, and $\wedge$,$\vee$ are binary operators on *S*.
- $\cup$ and $\cap$ are binary operators on the set of all sets.

One-to-One Functions

- A function is *one-to-one* (*1-1*), or *injective*, or *an injection*, if every element of its range has *only* 1 pre-image.
  - Formally: given $f{:}A\rightarrow B$, "f is injective" :≡ $(\neg\exists x,y\colon x\neq y \wedge f(x)=f(y))$.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
- For memorizing: each element of the domain is <u>injected</u> into a different element of the range.

One-to-One Illustration

- Bipartite graph representations of functions that are (or not) one-to-one:



One-to-one

Not one-to-one

Not even a function!

Sufficient Conditions for Injection

- For functions *f* over numbers, we say:
  - *f* is *strictly* (or *monotonically*) *increasing* if $x>y \rightarrow f(x)>f(y)$ for all *x*,*y* in domain;
  - *f* is *strictly* (or *monotonically*) *decreasing* if $x>y \rightarrow f(x)<f(y)$ for all *x*,*y* in domain;
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one.
- Examples

$f(x) = x^3$

$f(x) = -x^3$

Onto (Surjective) Functions

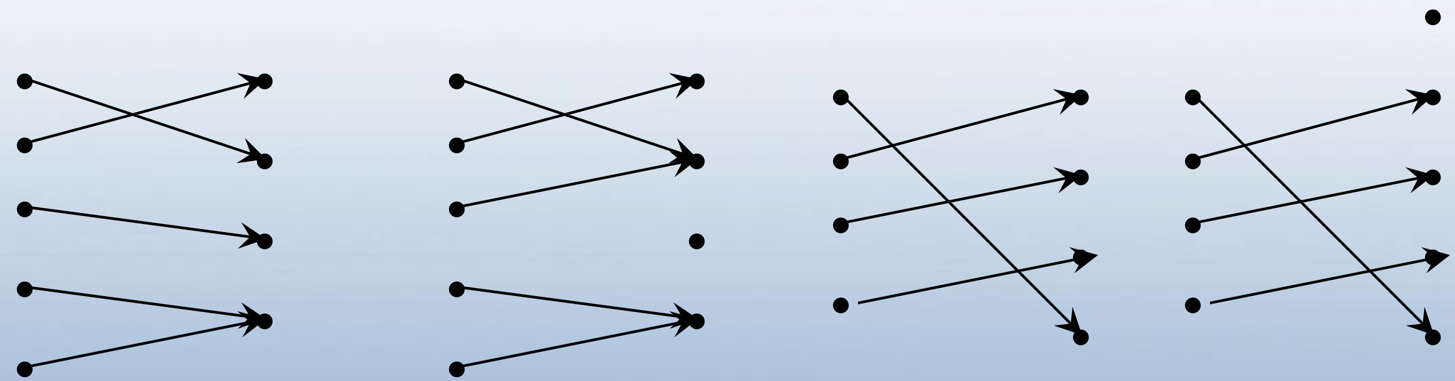
- A function f:$A \rightarrow B$ is *onto* or *surjective* or *a surjection* if its range is equal to its codomain ($\forall b \in B, \exists a \in A$: $f(a)=b$).

- An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.

- Example: Let f: R $\rightarrow$ R.
- $f(x) = x^3$ is surjective,
- $f(x) = x^2$ is not surjective. (Why?)

Illustration of Surjection

Some functions that are, or are not, *onto* their codomains:



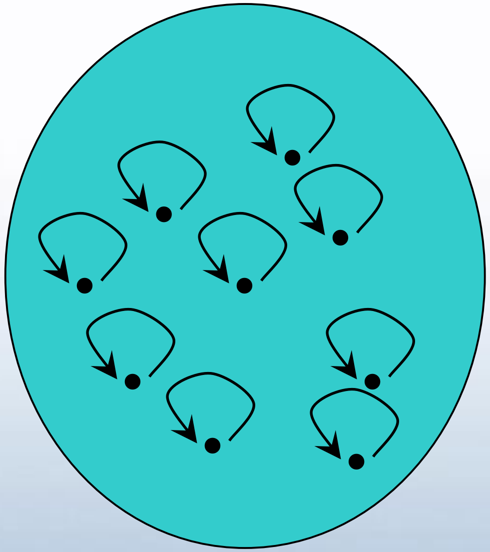
Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto

Identity Function

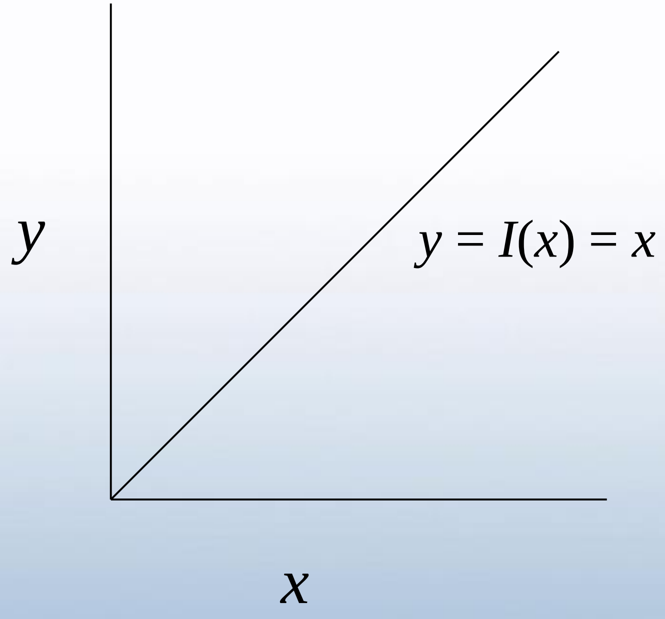
- For any domain *A*, the *identity function* $I:A \rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a \in A, I(a)=a$.

- Some identity functions you already know:
  - Summation of a number with 0, multiplication by 1,
  - conjunction with **True** value, disjunction with **False** value,
  - union with empty set $\varnothing$, intersection with universal set *U*.
- The identity function is always both one-to-one and onto.

Identity Function Illustrations



Domain and range



Bijections

- A function *f* is said to be *a bijection*, (or *a one-to-one correspondence*, or *reversible*, or *invertible*,) if it is both one-to-one and onto, both injective and surjective.
- For bijections $f{:}A \rightarrow B$, there exists an *inverse of f*, written $f^{-1}{:}B \rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$

  where I_A is the identity function on *A*

Some of Key Functions

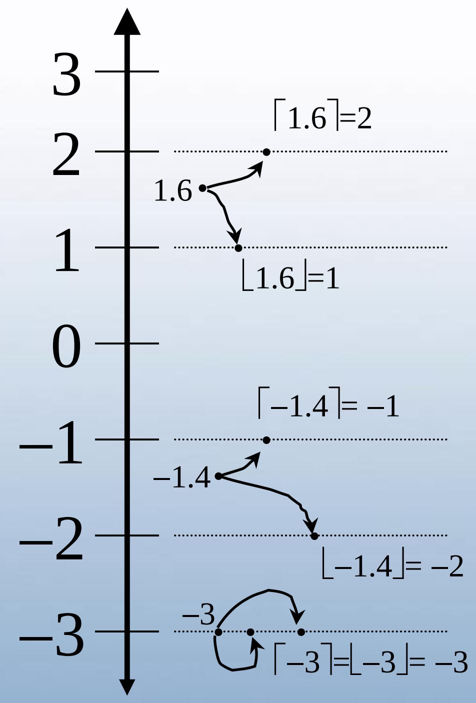
- In discrete math, we will frequently use the following two functions over real numbers:

- The *floor* function $\lfloor\cdot\rfloor$:R→Z, where $\lfloor x \rfloor$ ("floor of *x*") means the largest integer $\leq x$. *Formally,* $\lfloor x \rfloor :\equiv \max(\{i \in \mathbf{Z} \mid i \leq x\})$.

- The *ceiling* function $\lceil\cdot\rceil$:R→Z, where $\lceil x \rceil$ ("ceiling of *x*") means the smallest integer $\geq x$. *Formally,* $\lceil x \rceil :\equiv \min(\{i \in \mathbf{Z} \mid i \geq x\})$

Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ and $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.

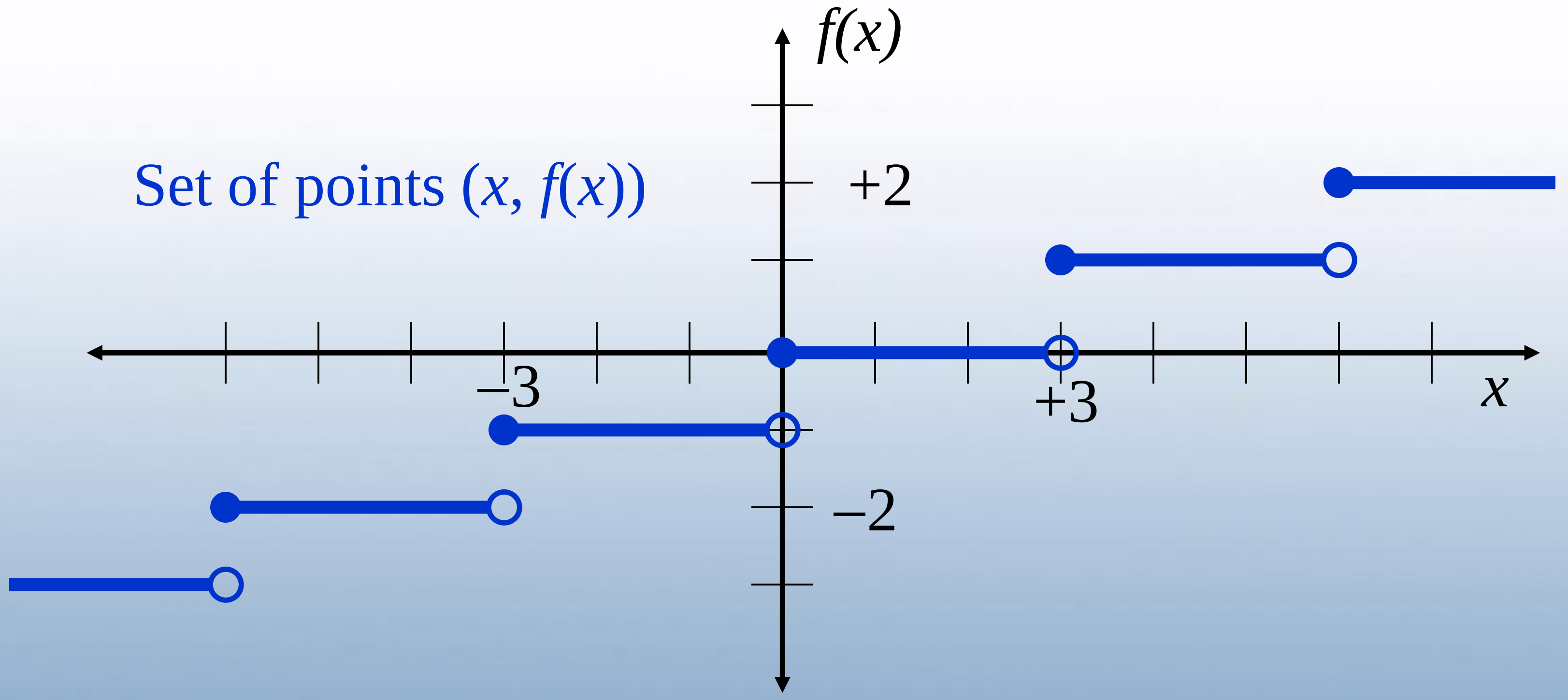


Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of Functions

- Notations: $f{:}A \rightarrow B$, $f(a)$, $f(A)$.
- Terms:
  image, preimage, domain, codomain,
  range, one-to-one, injection,
  onto, sujection, bijection
- Inverse function f^{-1} and identity function I_A
- $\mathbf{R} \rightarrow \mathbf{Z}$ functions $\lfloor x \rfloor$ and $\lceil x \rceil$.

How do we define sets?

Are functions useful here?