Geometric Modeling

Function Representation FRep

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Motivation

- Links of “implicits” with other models
- Single real-valued function
- Solid sweeps
- Blends in CAGD
- Deformations
- Collision detection
- Metamorphosis

Motivation: Links With Other Models

- **CSG**: algebraic (quadric) primitives of CSG;
  algebraic patches in constructive shells;
  ray-tracing algorithms.
- **BRep**: algebraic patches;
  polygonization.
- **Sweeping**:
  generalized cylinders with skeletal curves.
- **Voxels**: polygonization algorithms.

Motivation: Single Real-valued Function

The idea of representation of an **entire** complex object by a single real function:

- applied effectively in **skeletal implicits**;
- **min/max** operations for CSG proposed by Ricci [1973] have not found wide acceptance (C^1 discontinuity as a reason);
- existence of **R-functions** proposed by Rvachev [1963], surveyed by Shapiro [1988], and applied in multidimensional geometric modeling by Pasko [Ph.D. thesis, 1988].

More deep connection with *CSG* is possible.

Motivation:
Solid Sweeps

- Theoretical possibility to derive an implicit description of a **surface swept by a woving solid** [Wang 1984].
- Symbolic computations required to yield a formula for the implicit form.

More deep connection with *sweeping* is possible.

Motivation: Blends in CAGD

- Attention is paid in Computer Aided Geometric Design (CAGD) to implicit surfaces because of their closure under some important operations: **offsetting** and **blending** [Hoffman 1993].

Motivation: Deformations and Collisions

Deformations available for “implicits”:

- Twist, bend, taper [Barr]
- Free vibrations [Sclaroff and Pentland 1991].

Collision detection algorithms for implicit surfaces [Gascuel 1993]

Motivation:
Voxels as Discrete Fields

- Similar **polygonization** algorithms stress common nature of implicit and voxel models.
- **Metamorphosis** of skeletal implicits [Wyvill 1991] and scheduled Fourier volume morphing [Hughes 1992].

More deep connection with *voxel models* is possible.

Motivation: Survey Conclusion

- Representations by real-valued functions are widely used in geometric modeling and computer graphics in several forms.
- These models are not closely related to each other.
- These models are not closely related to such well-known representations as CSG, B-rep, sweeping, and spatial partitioning (voxel models).

Motivation: Survey Conclusion

- This obviously retards further R&D.
- A uniform *function representation* is needed to fill these gaps.
- This representation has to:
  - unify all functionally based approaches;
  - be convertible from other representations;
  - be dimension independent;
  - have as rich as possible a system of operations and relations.

FRep: Geometric Concepts

Geometric concepts of a functionally based modeling environment:

$$(\boldsymbol{M}, \boldsymbol{\Phi}, \boldsymbol{W})$$

where

M is a set of *geometric objects*,

Φ is a set of *geometric operations*,

W is a set of *relations* on the set of objects.

Mathematically this triple is a sort of ***algebraic system***.

Objects

Closed subsets of *n*-dimensional Euclidean space E^n with the definition:

$$F(x_1, x_2, ..., x_n) \geq 0$$

where *F* is a real-valued continuous function defined on E^n.

Objects

⇨ Classification of points in E^n space:

$F(\boldsymbol{X}) > 0$ - for points inside the object;

$F(\boldsymbol{X}) = 0$ - for points on the object's boundary;

$F(\boldsymbol{X}) < 0$ - for points outside the object.

Here, $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)$ is a point in E^n.

Objects

- ⇨ In 3D space, the boundary of such an object is a so-called ***"implicit" surface***.
- ⇨ The function can be defined by:
  1) analytical expression;
  2) function evaluation algorithm;
  3) tabulated values and an appropriate interpolation procedure.
- ⇨ The major requirement to the function is to have at least **C^0 continuity**

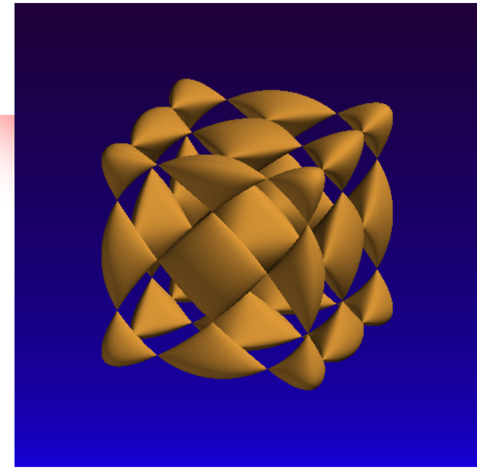
Objects

- ⇨ The inequality with the **explicit function of *n* variables**, but **not the implicit function** of *n*-1 variables $f(x_1, x_2, ..., x_n) = 0$.
- ⇨ **Multidimensional** formulation. $n = 4$: space-time modeling.
- ⇨ ***Primitives*** and ***complex objects***. A complex geometric object is a result of operations on primitives.
- ⇨ **Extensibility**

Types of Primitives

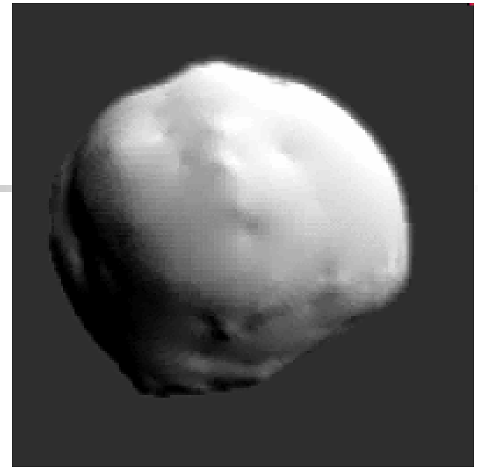
- Algebraic surfaces
- Skeletal objects (soft, blobby, etc.)
- Convolution surfaces
- Radial-basis functions (volume splines)
- Trivariate B-splines
- Voxel array + interpolation
- Procedural

Primitives

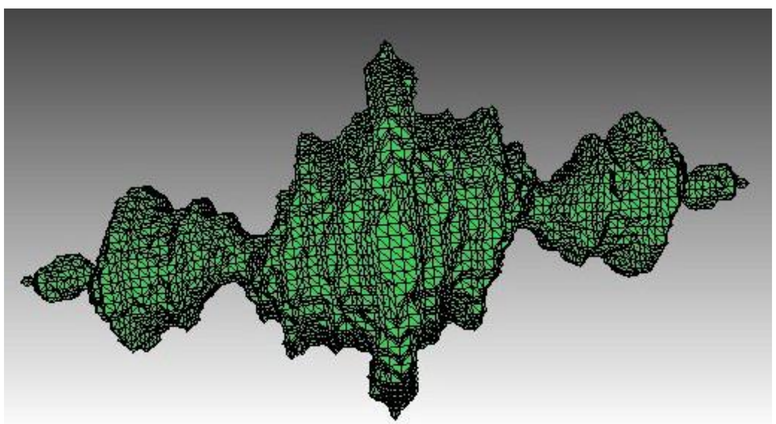


Algebraic

Voxel array



Radial-basis RBF



Procedural fractal

Trivariate B-spline

Complex Object: Single Function

Operations

$$\Phi_i: M^1 + M^2 + \ldots + M^n \rightarrow M$$

⇨Two main classes:

1) ***unary*** (*n=1*) operations

$G_2 = \square_i(G_1)$

$f_2 = \square\ (f_1(\mathbf{X}))\ \square\ 0$

2) ***binary*** *(n=2) operations*

$G_3 = \square_i(G_1, G_2)$

$f_3 = \square\ (f_1(\mathbf{X}), f_2(\mathbf{X}))\ \square\ 0$

Relations

A binary relation is a subset of the set $M^2 = M \times M$. It can be defined as

$$S_i: M \times M \rightarrow I$$

The examples of binary relations are inclusion, point membership, interference or collision.

Point Membership Relation

$$S_3(P, G_1) = \begin{cases} 0, & \text{if } f_1(\mathbf{X}) < 0 \quad \text{for } P \notin G_1 \\ 1, & \text{if } f_1(\mathbf{X}) = 0 \quad \text{for } P \in bG_1 \\ 2, & \text{if } f_1(\mathbf{X}) > 0 \quad \text{for } P \in iG_1 \end{cases}$$

P is a point

iG_1 is the interior of G_1

bG_1 is the boundary of G_1.

Unary Operations

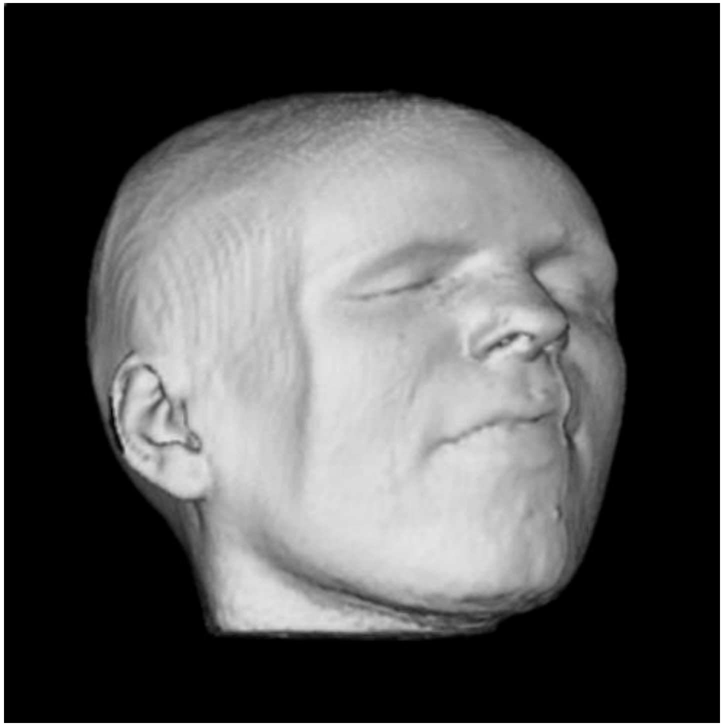
- Bijective space mapping (deformations)
- Offsetting
- Projection
- Sweeping

Deformations: Twisting

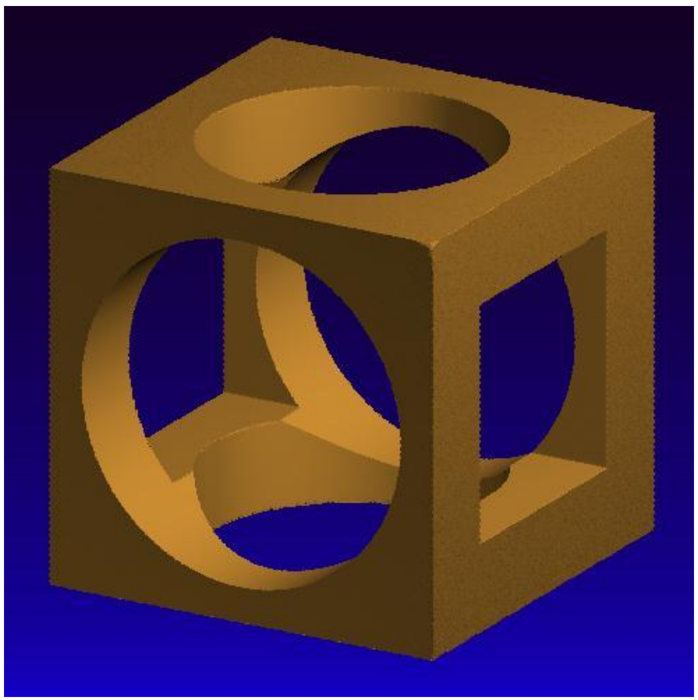
Feature-based mapping

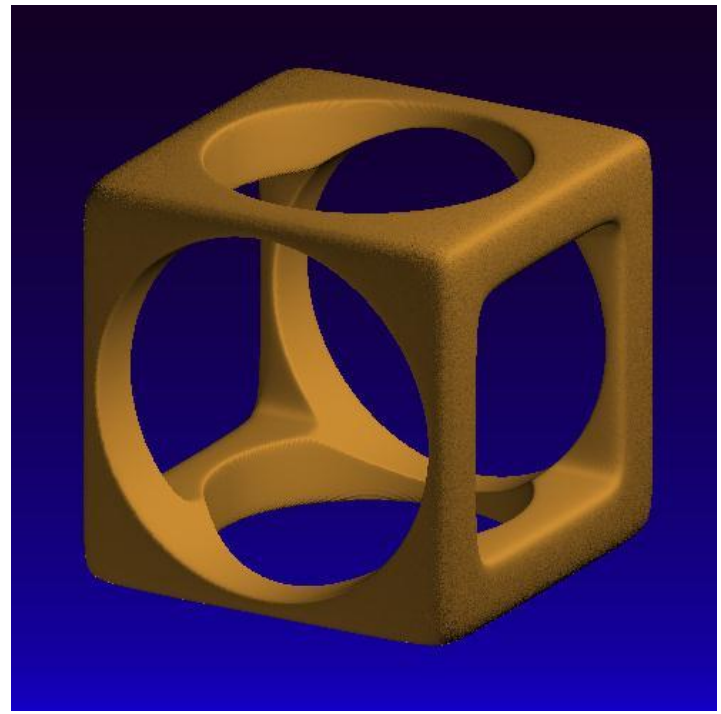
 

Feature-based mapping

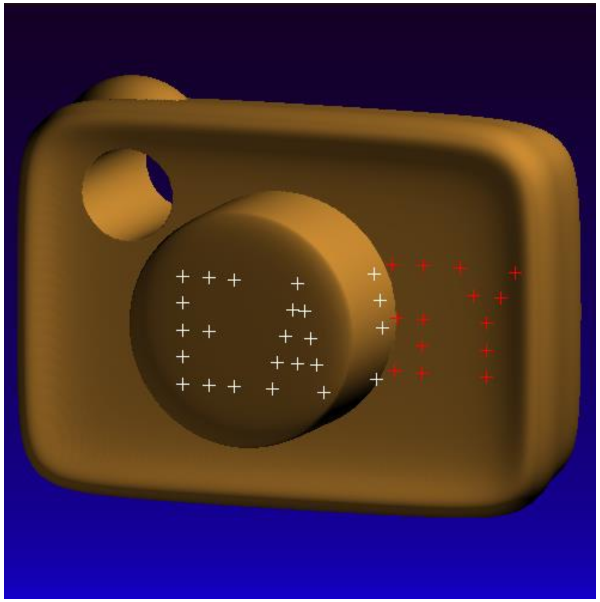


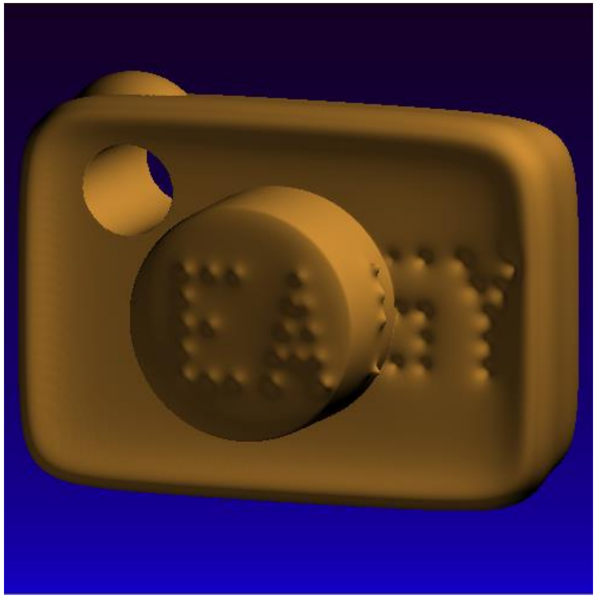
Unary Operations: Offsetting



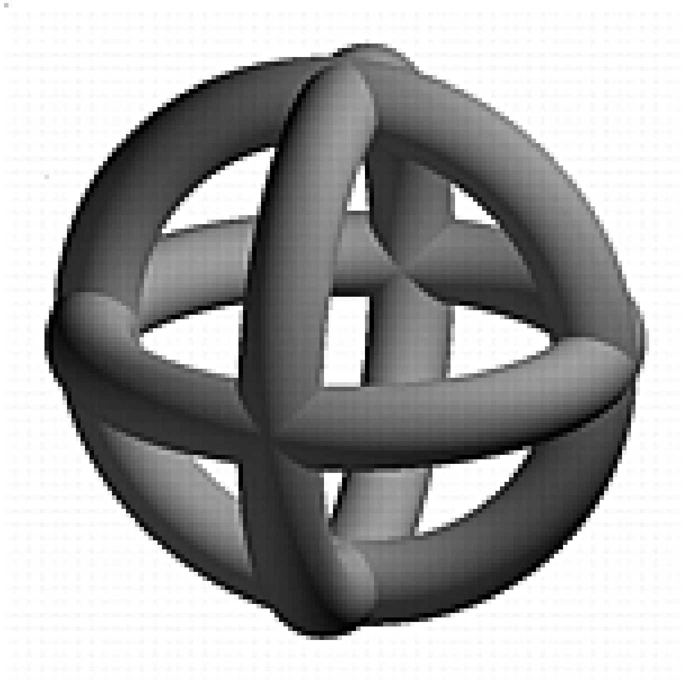


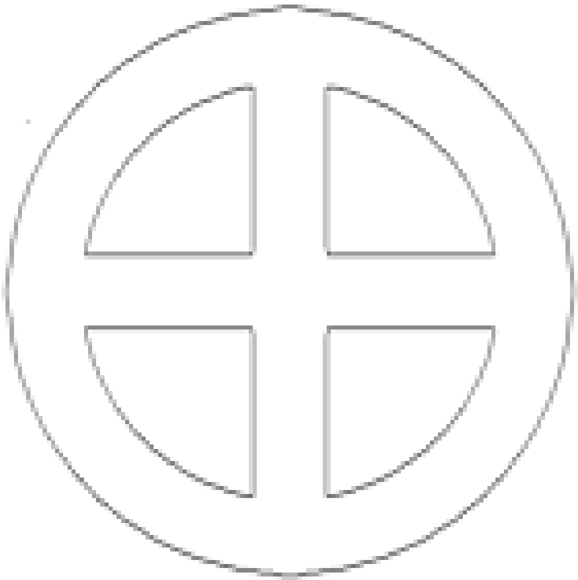
Feature-based offsetting



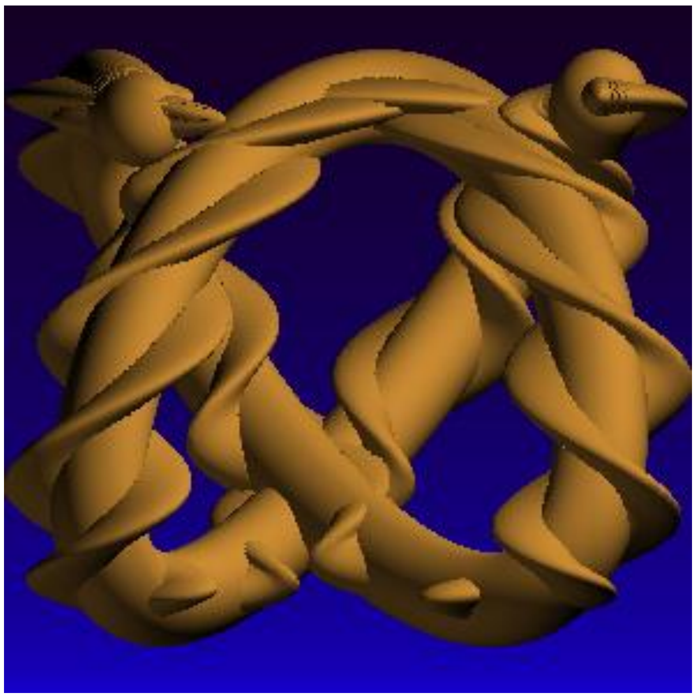


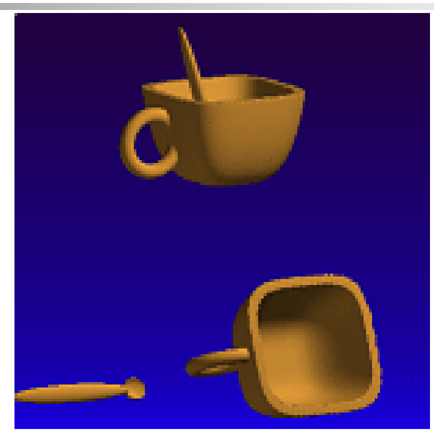
Unary Operations: Projection

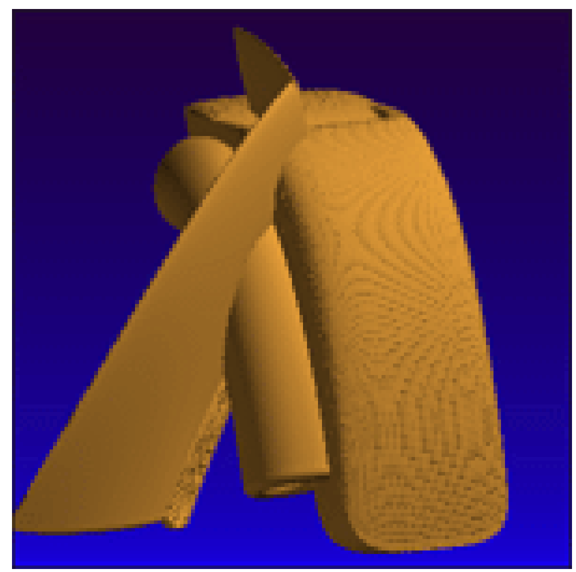




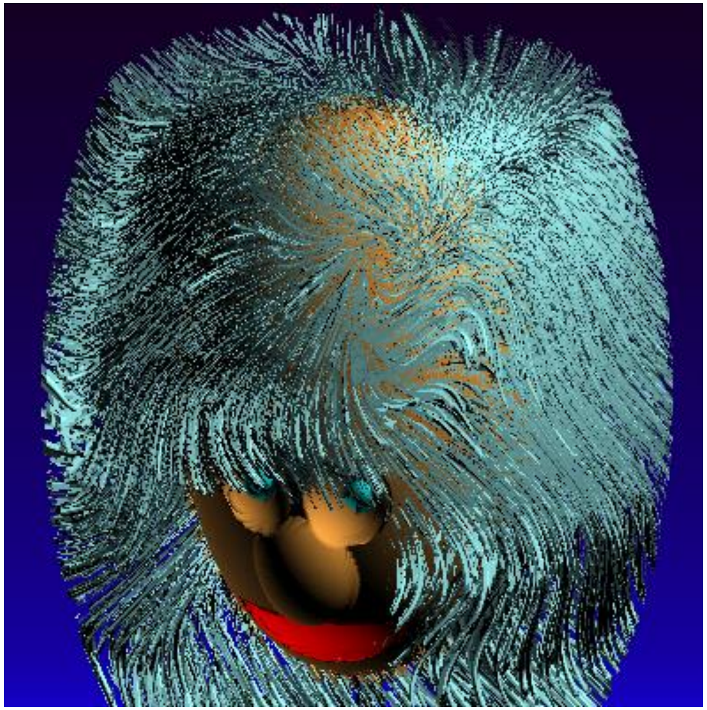
Unary Operations: Sweeping

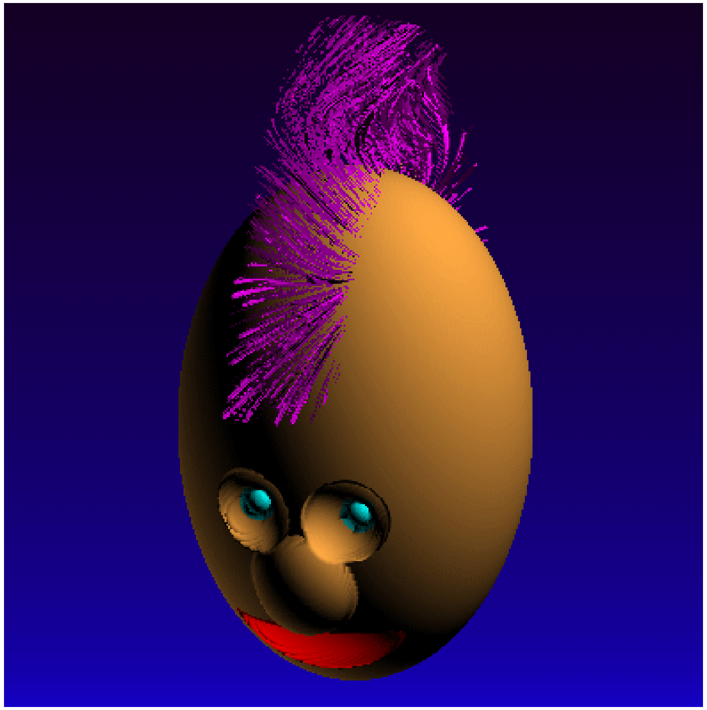






Unary Operations: Hypertexturing



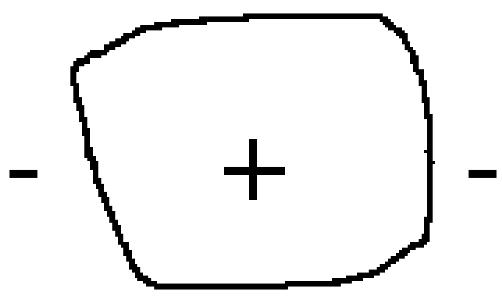


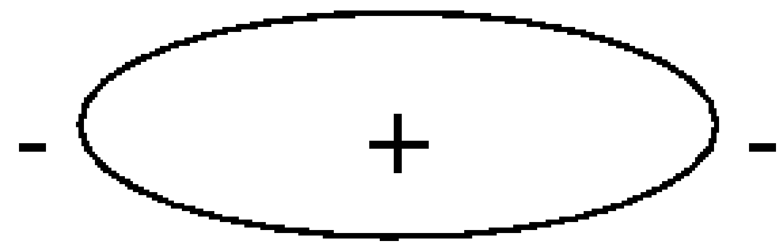
Binary Operations

- Set-theoretic operations:
  Intersection
  Union
  Difference
- Blending
- Metamorphosis

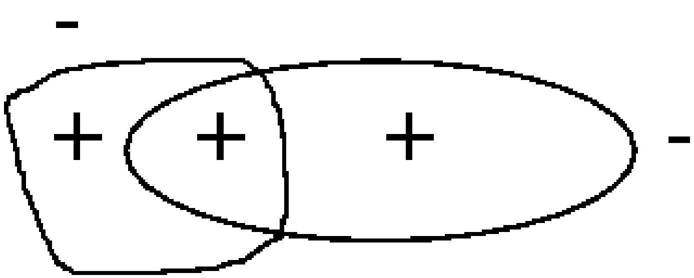
Set-theoretic Operations

$G_1 : f_1(\boldsymbol{X}) \geq 0$ $\qquad$ $G_2 : f_2(\boldsymbol{X}) \geq 0$

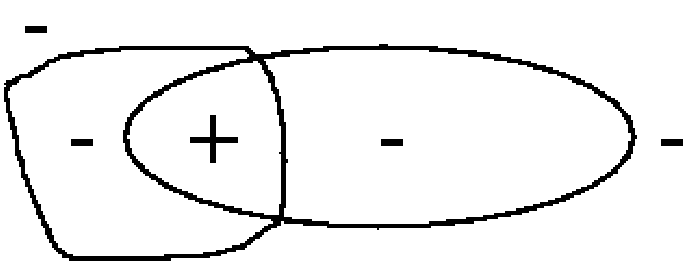




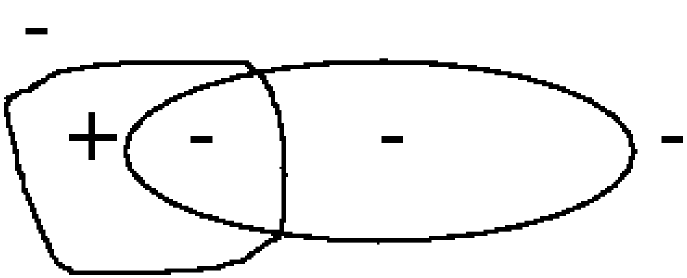
Set-theoretic Operations



$$G_3 = G_1 \cup G_2$$



$$G_3 = G_1 \cap G_2$$



$$G_3 = G_1 \setminus G_2$$

$$f_3 = \Psi\,(f_1(\boldsymbol{X}), f_2(\boldsymbol{X})) \quad ?$$

Types of R-functions

Union $G_3 = G_1 \cup G_2 \rightarrow f_3 = f_1 \mid f_2$

Intersection $G_3 = G_1 \cap G_2 \rightarrow f_3 = f_1 \,\&\, f_2$

Subtraction $G_3 = G_1 \setminus G_2 \rightarrow f_3 = f_1 \setminus f_2$

|,**&** ,\ are signs of R-functions.

Types of R-functions

Rvachev [1972]

$$f_1 | f_2 = \frac{1}{1+\alpha}\left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2}\right)$$

$$f_1 \& f_2 = \frac{1}{1+\alpha}\left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2}\right)$$

$$f_1 \setminus f_2 = f_1 \& (-f_2)$$

$$-1 < \alpha(f_1, f_2) \le 1,$$

$$\alpha(f_1, f_2) = \alpha(f_2, f_1) = \alpha(-f_1, f_2) = \alpha(f_1, -f_2)$$

Types of R-functions

If $\alpha=0$, the above functions take the most useful in practice form:

$$f_1 \mid f_2 \quad = f_1 + f_2 + \sqrt{f_1^2 + f_2^2}$$

$$f_1 \,\&\, f_2 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}$$

These functions have C^1 discontinuity only in points where both arguments are equal to zero.

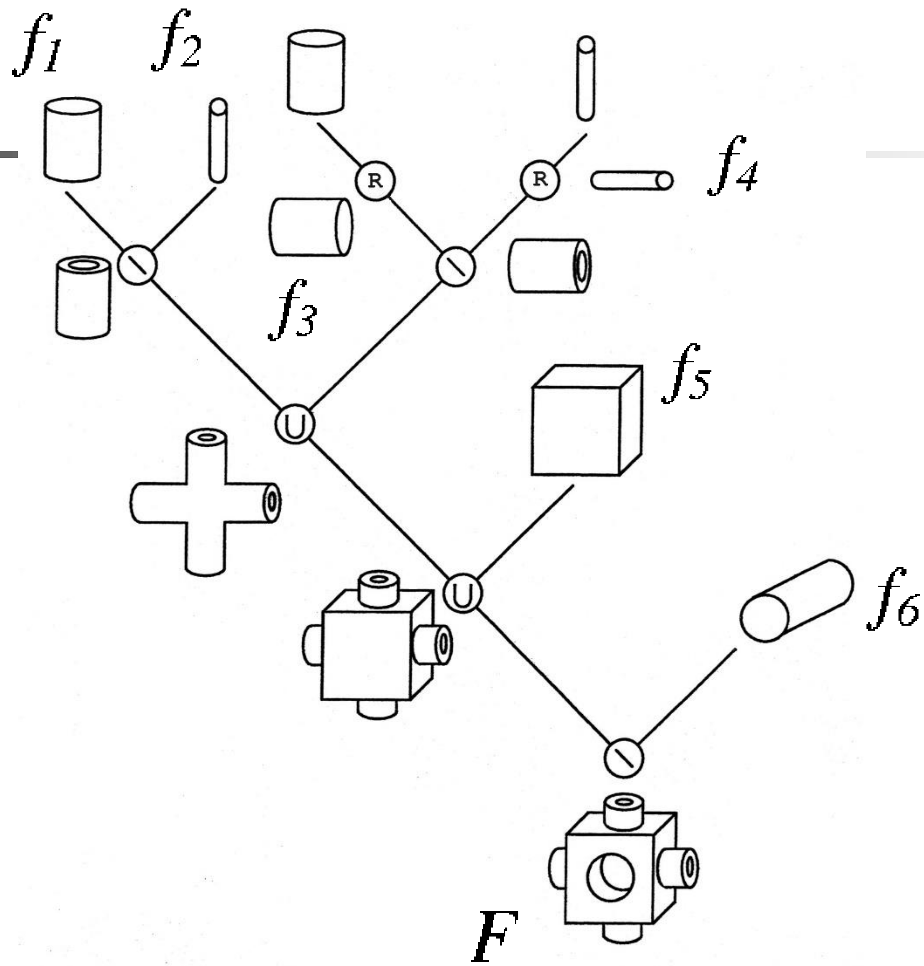
Types of R-functions

If C^m continuity is to be provided, one may use another set of R-functions:

$$f_1 | f_2 = \left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2}\right)\left(f_1^2 + f_2^2\right)^{\frac{m}{2}}$$

$$f_1 \& f_2 = \left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2}\right)\left(f_1^2 + f_2^2\right)^{\frac{m}{2}}$$

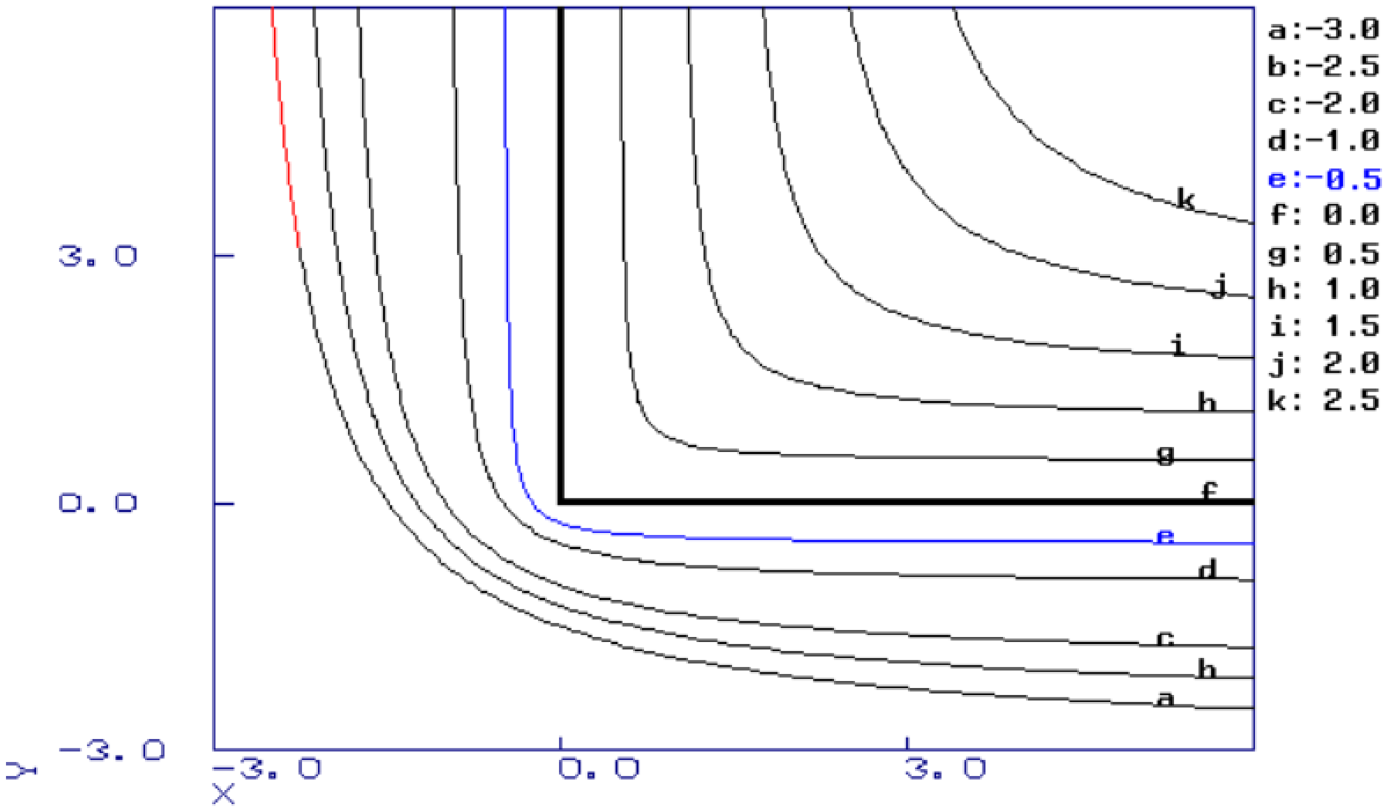
Example



$$F = (((f_1 \setminus f_2) \mid (f_3 \setminus f_4)) \mid f_5) \setminus f_6$$

Contours of R-functions

$$f = x + y - \sqrt{x^2 + y^2}$$

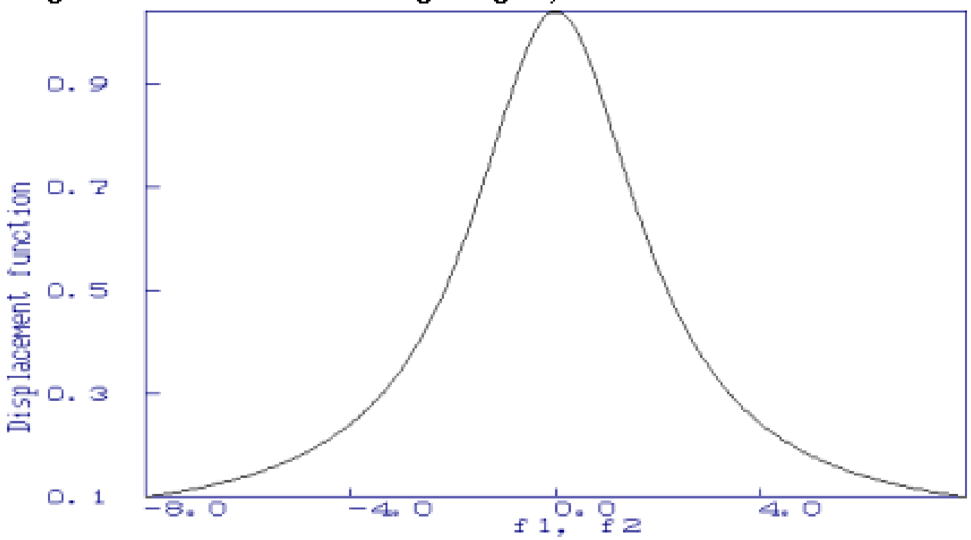


Blending

$$F(f_1,f_2) = R(f_1,f_2) + d(f_1,f_2)$$

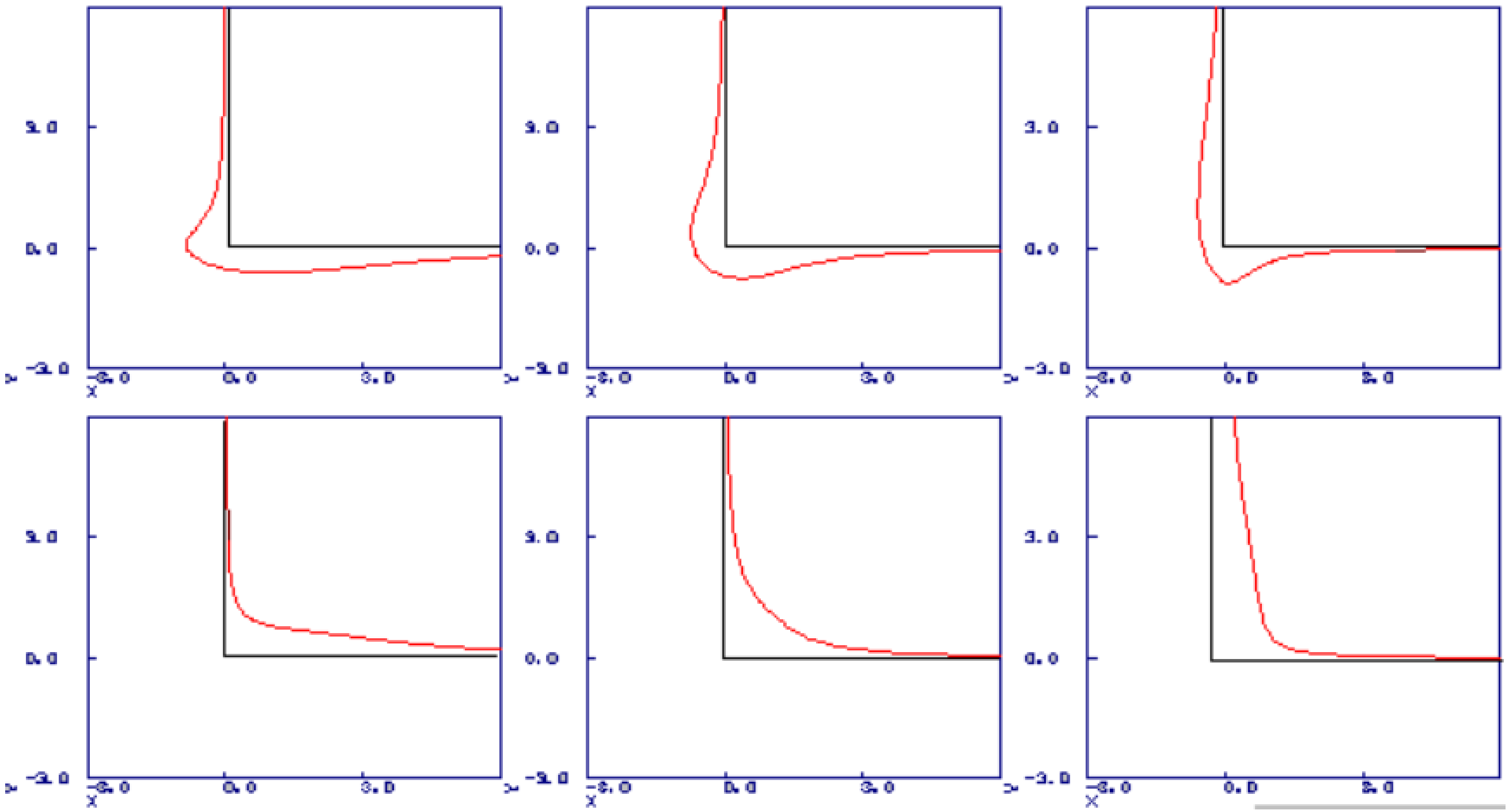
$$d(f_1,f_2) = a_0/(1+(f_1/a_1)^2+(f_2/a_2)^2)$$

The section $f_1=0$ for $d(f_1,f_2)$:

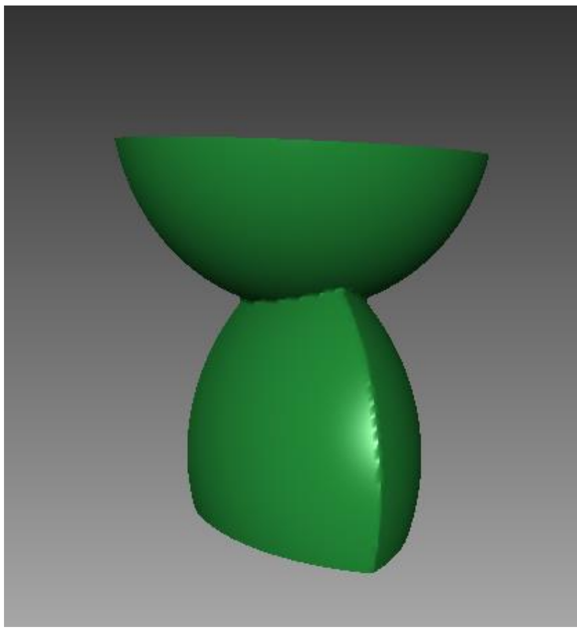


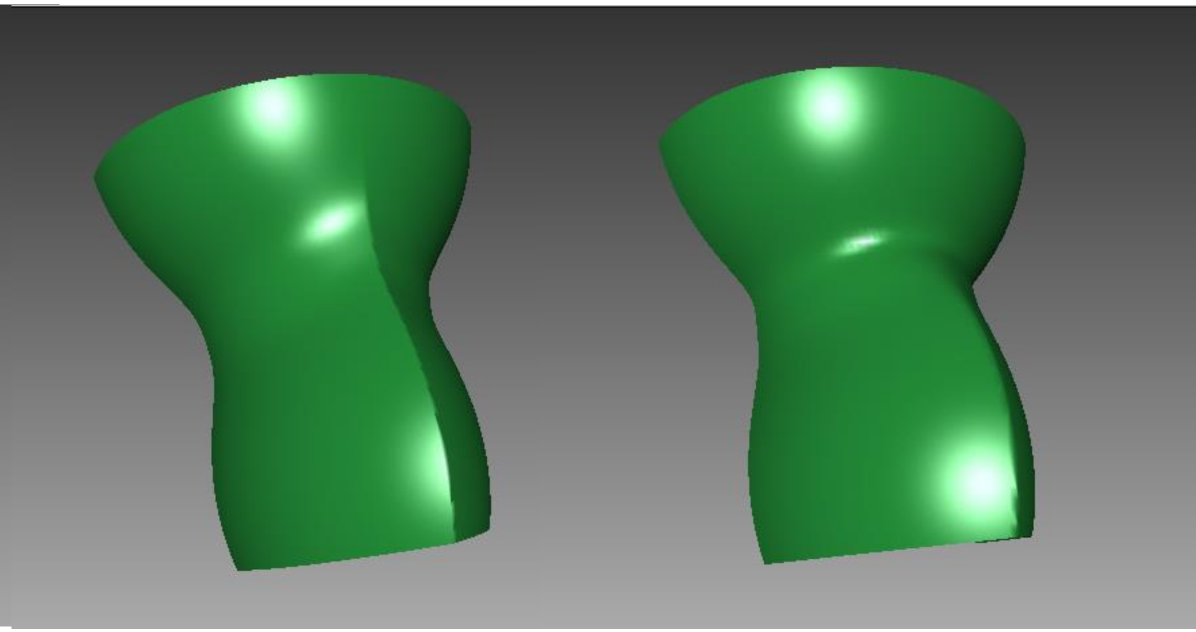
Blending Intersection

$$F(f_1,f_2) = f_1 + f_2 - \sqrt{f_1^2 + f_2^2} + a_0/(1+(f_1/a_1)^2+(f_2/a_2)^2)$$



Blending to an edge





Union Min/max R-function

Blending Union

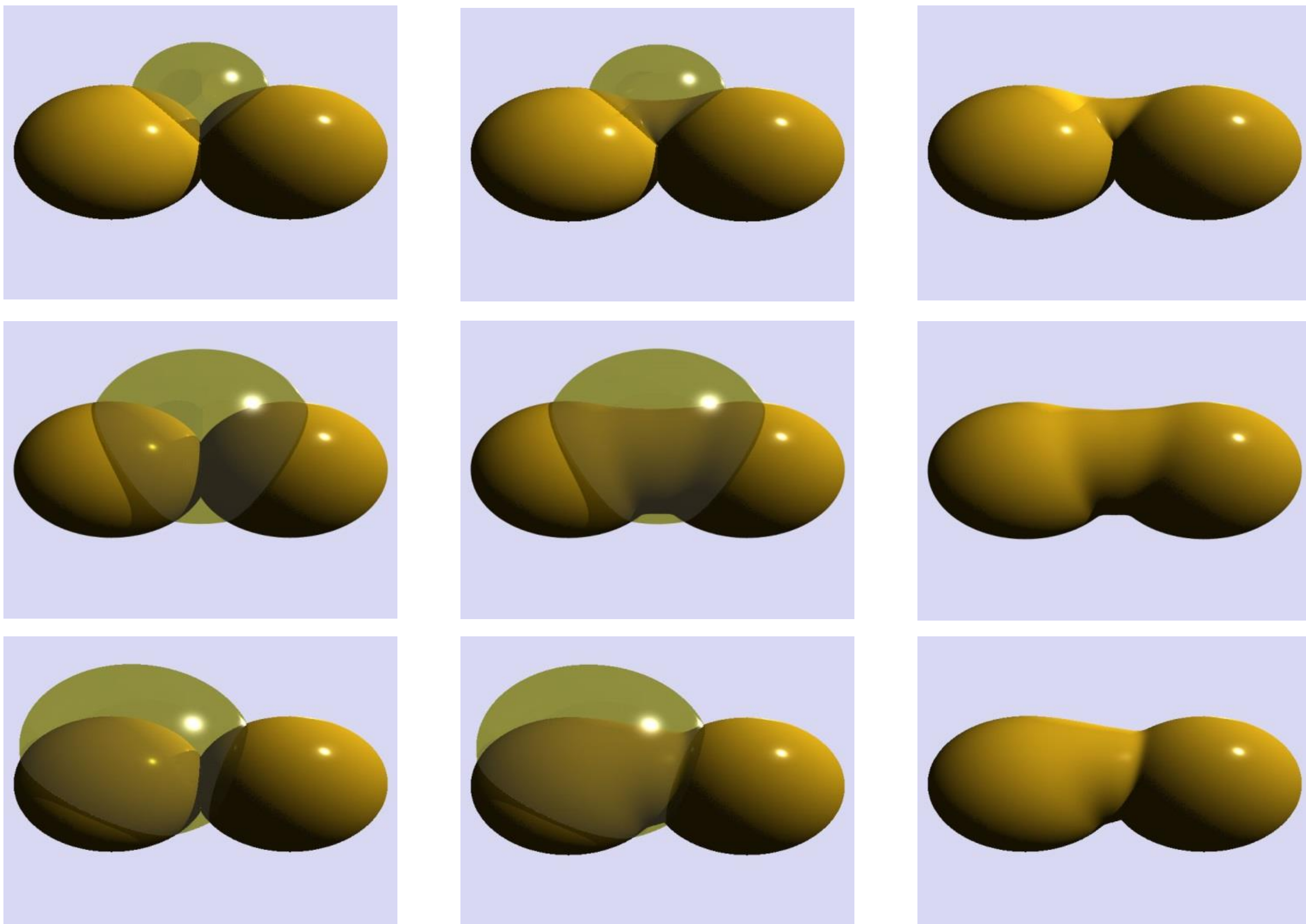
Bounded displacement function

1) $disp_{bb}(r)$ takes the maximal value for $r=0$;

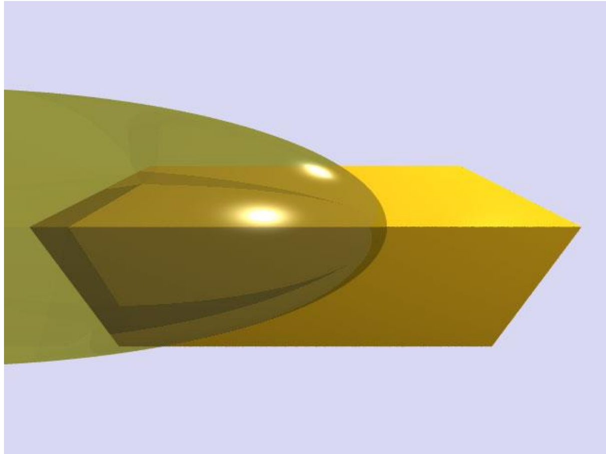
2) $disp_{bb}(r) = 0, r \geq 1$

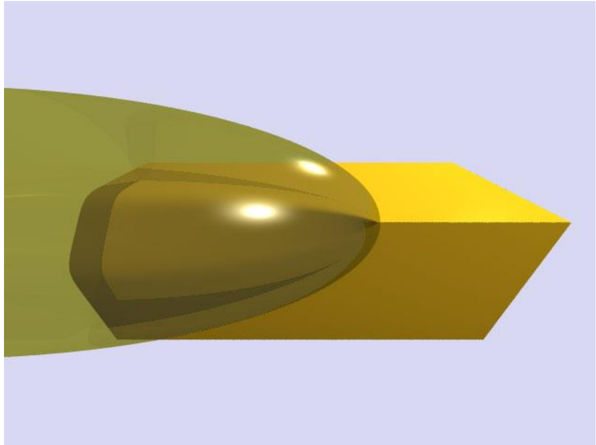
3) $\frac{\partial disp_{bb}}{\partial r} = 0, r = 1$ the curve tangentially approaches the axis at $r=1$.

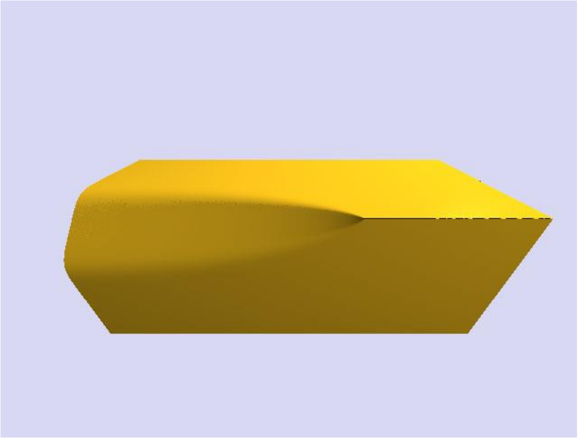
Bounded Blending



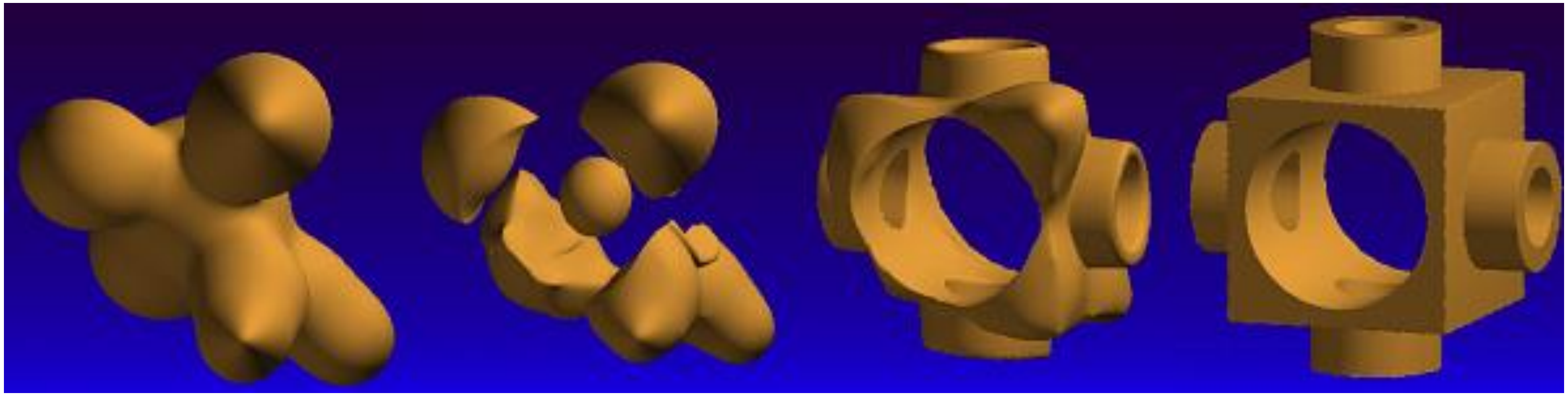
Partial edge blending



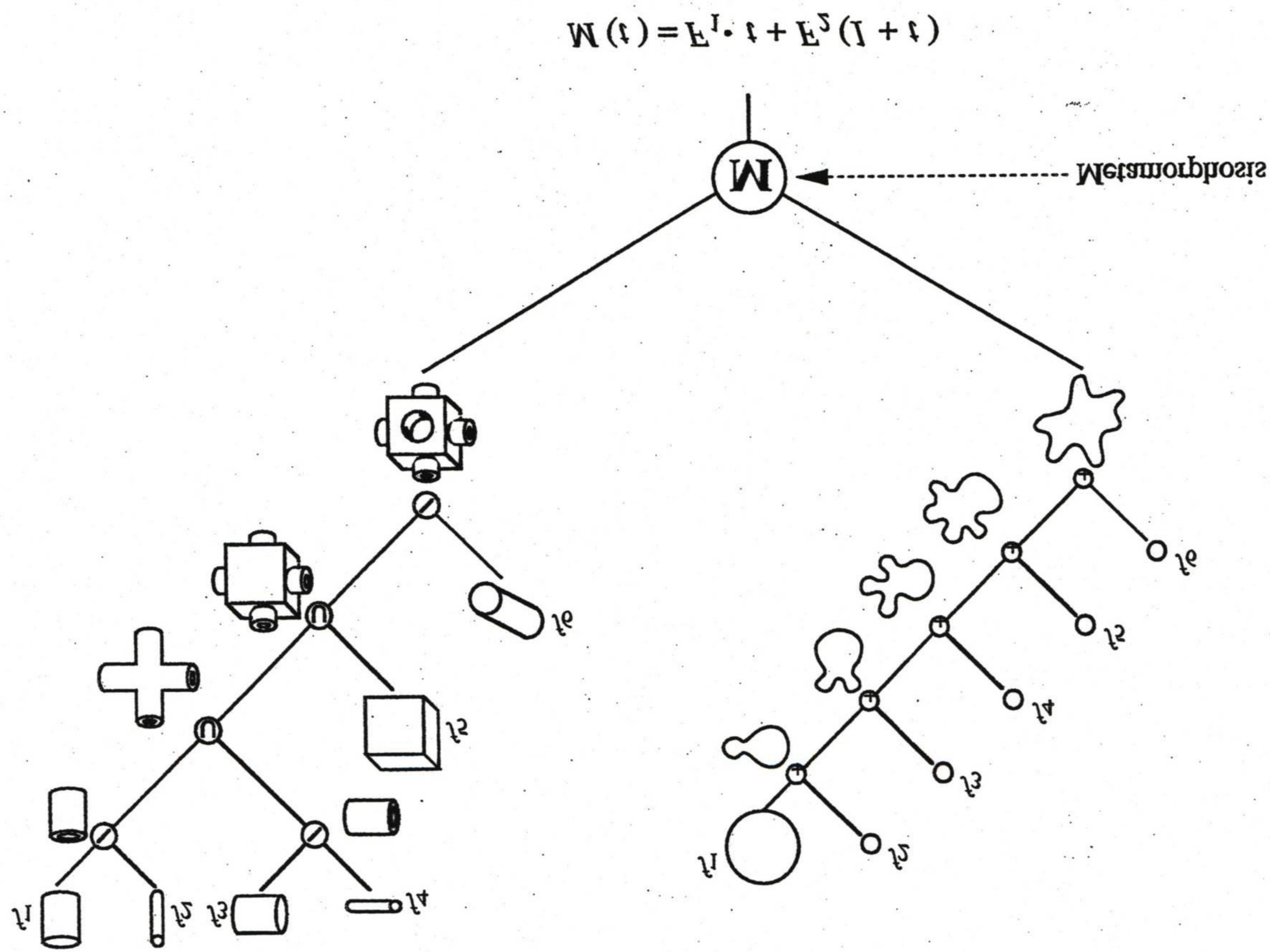




Metamorphosis



Construction tree



What is FRep, anyway?

- Uniform represenation of multidimensional solids defined as

$$F(X) \geq 0$$

- Function $F(X)$ evaluation procedure traversing the construction tree structure
- Leaves: <u>primitives</u>
- Nodes: <u>operations</u> + <u>relations</u>
- "Empty Case" principle and extensibility

Software Tools

Specific forms:

- SvLis [Bowyer et al.]
- BlobTree [B. Wyvill et al.]
- MeshUp [Uformia]

Full FRep:

- HyperFun [Adzhiev et al.]
- Symvol for Rhino [Uformia]

Heterogeneous Objects

- Internal structure with non-uniform distribution of material and other attributes
- entities of different dimensionalities
- varying material distribution in CAD/CAM and rapid prototyping
- physical simulations, geological and medical modeling, volume modeling and rendering

⇨ **Need for a unifying model**

Constructive Hypervolume Model

$$o = (\ F(X),\ S_1\ (X),\ \ldots,\ S_k\ (X)\ )$$

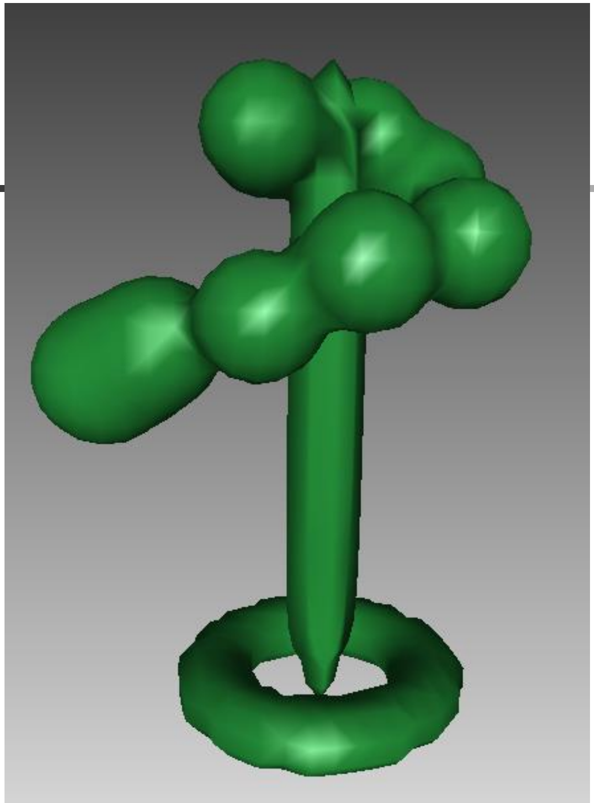
$F(X)$ - FRep of geometry

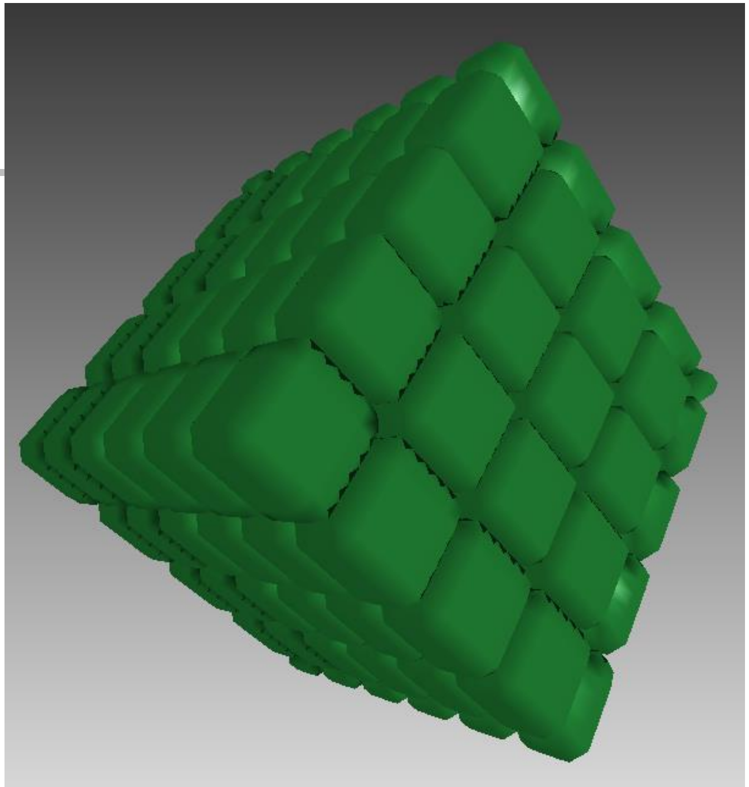
$S_i\ (X)$ - FRep of attribues

Constructive Hypervolume Model

- Point set geometry and attributes (physical, photometric, etc.): independent representation but uniform treatment
- Constructive modeling of both geometry and attributes using primitives, operations and relations
- Using real-valued functions (scalar fields)

Constructive Hypervolume Model

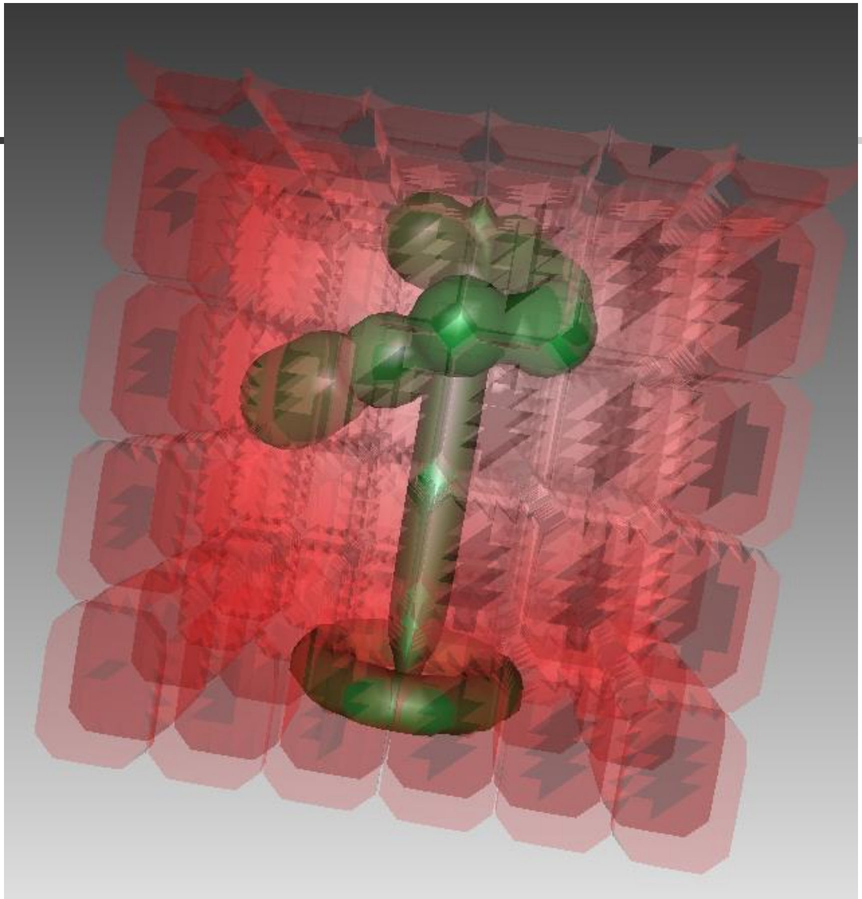


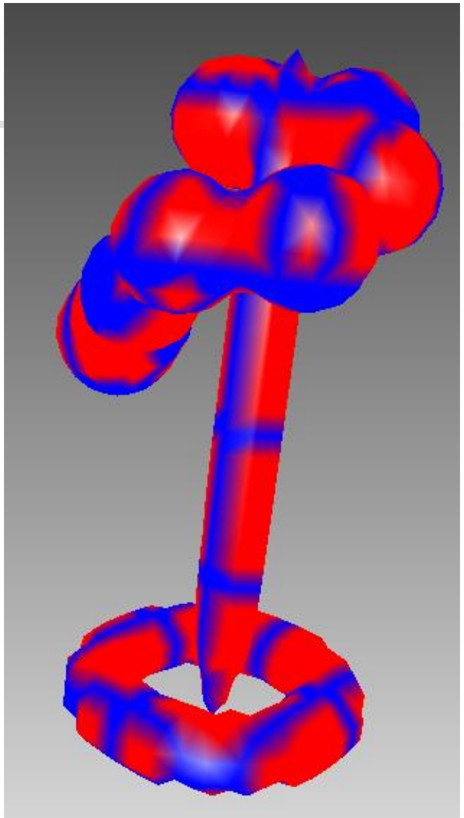


Geometry

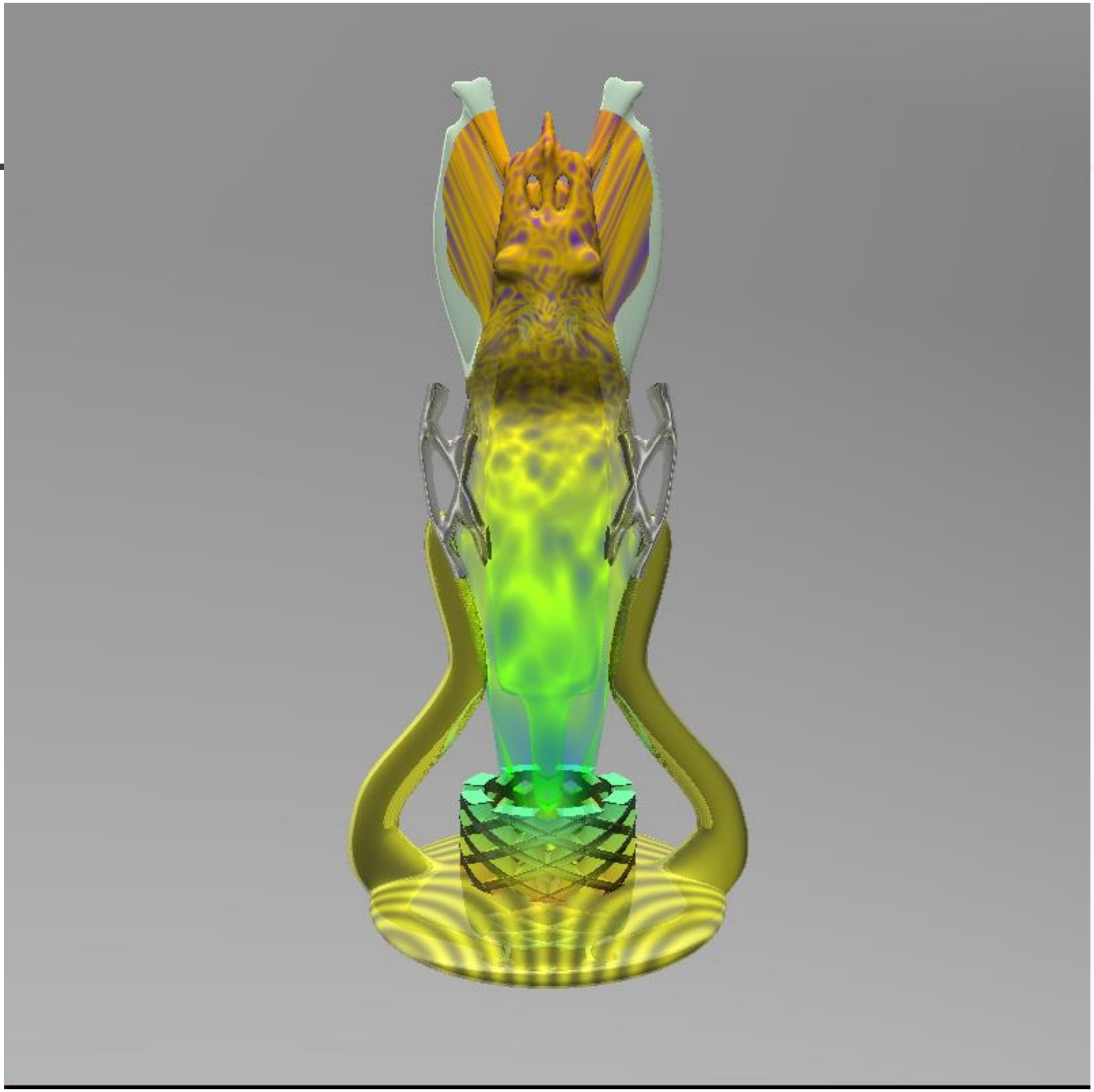
Attribute
space partitioning

Constructive Hypervolume Model





Constructive Hypervolume Model



General References

- Pasko A., Adzhiev V., Sourin A., Savchenko V., **Function representation in geometric modeling: concepts, implementation and applications**, The Visual Computer, vol. 11, No. 8, 1995, pp. 429-446.

General References

- Rvachev V.L., **On the analytical description of some geometric objects**, Reports of Ukrainian Academy of Sciences, vol. 153, No. 4, 1963, pp. 765-767 (in Russian).
- Ricci A., **A constructive geometry for computer graphics**, The Computer Journal, vol. 16, No. 2, 1973, pp. 157-160.

More References

- Savchenko V., Pasko A., **Transformation of functionally defined shapes by extended space mappings**, The Visual Computer, vol. 14, No. 5/6, 1998, pp. 257-270.
- Sourin A., Pasko A., **Function representation for sweeping by a moving solid**, IEEE Transactions on Visualization and Computer Graphics, vol. 2, No. 1, 1996, pp.11-18.
- Pasko A. et al., **Constructive hypervolume modeling**, Graphical Models, special issue on Volume Modeling, vol. 63, No. 6, 2001, pp. 413-442.