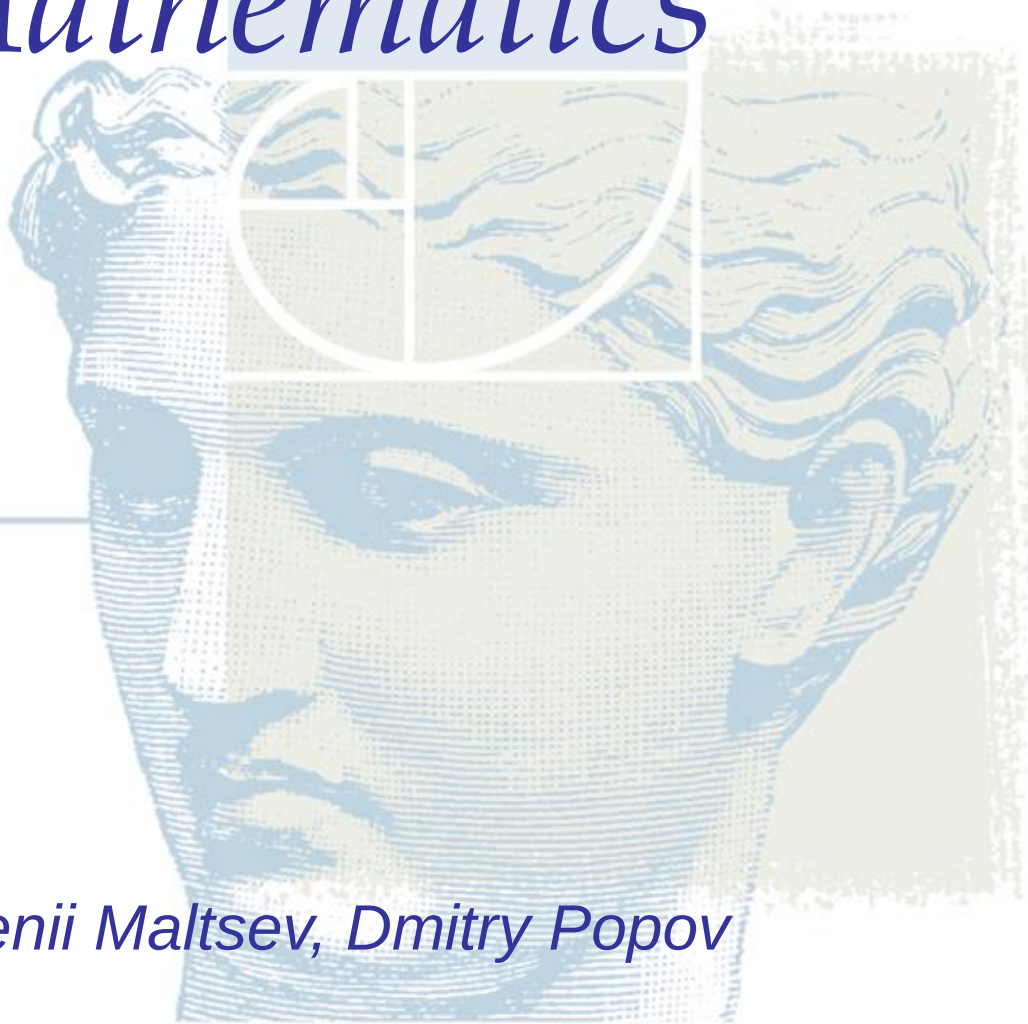
# *Discrete Mathematics*



*Alexander Pasko, Evgenii Maltsev, Dmitry Popov*

# Personal info

See www.pasko.org/ap for

Function Representation (FRep)
in procedural volume modelling

&

HyperFun project
(from Hyperdimensional Functions)

# Unit organization

- 20 lectures and seminars
- Exams at the end: some theory questions and practical problems to solve – attend seminars and keep materials
- *Repetition is the mother of learning*
- Connections to art and CG programming

# Unit materials

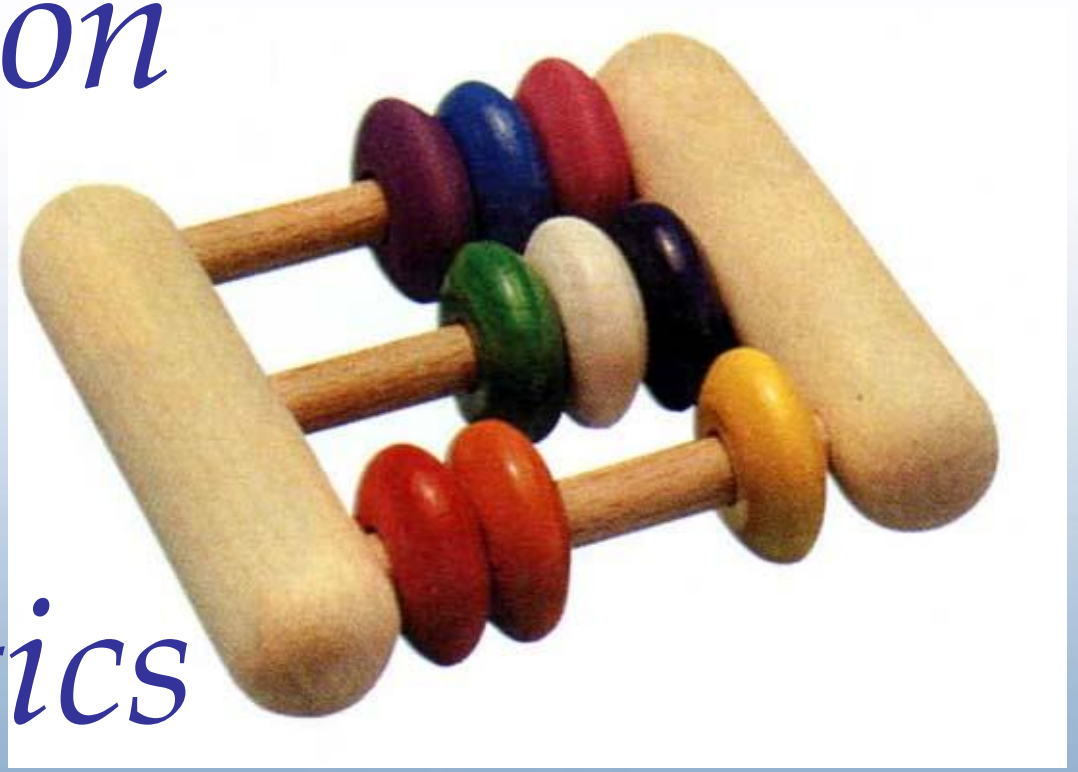
- Lecture notes
- Seminar handouts

are available at

http://gm.softalliance.net/

Advice: download and print lecture notes before the next lecture

# *Introduction to Discrete Mathematics*



# Contents

- Digital vs analog = discrete vs continuous
- Subject of discrete mathematics
- Some discrete structures
  & fields of discrete mathematics
- Why study discrete math?

# Analog and digital information

Two ways of representing information: **analog** and **digital**. It depends on type of variation in physical changes:

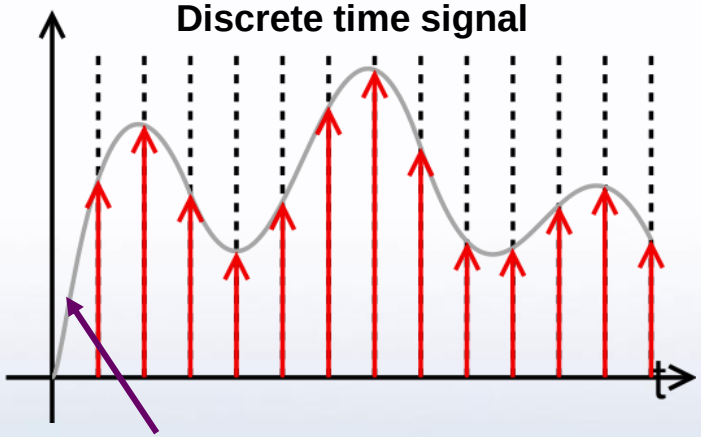
- If variation of changes is continuous, then message is said to be analog.
- If information in message is represented by variations that go in distinct steps, then message is digital.


# Signal categories

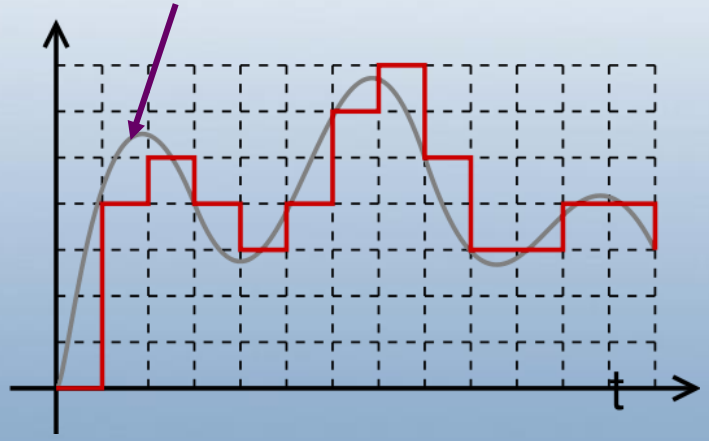
- Signal is any time-varying quantity
- Signal categories:

 - discrete time and continuous time

 - discrete-valued and continuous-valued



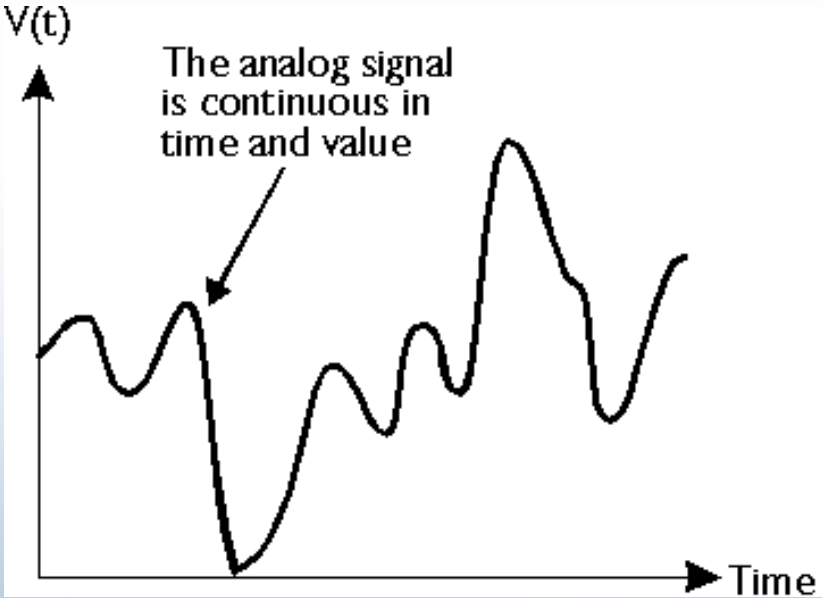

Continuous-valued and continuous time signal

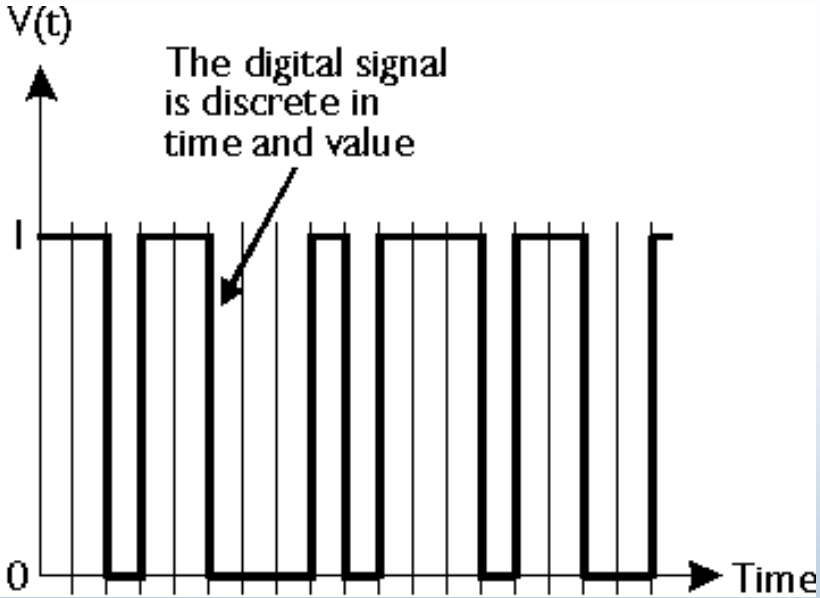



Discrete-valued and discrete time signal

http://en.wikipedia.org/wiki/Discrete_signal

# Analog and digital signals




Digital signals are discrete-valued, but derived from underlying analog physical processes.

www-scm.tees.ac.uk/.../DSP/analog_D.gif

# Examples of signals

- *Sound* - vibration of medium (air), sound signal associates pressure value to every value of time and three space coordinates (4D domain).
- *Picture* assigns color value to each of set of points.
  Since points lie on a plane, domain is 2D.
  - If picture is a physical object, it's continuous signal.
  - If picture is a digital image, it's discrete signal.

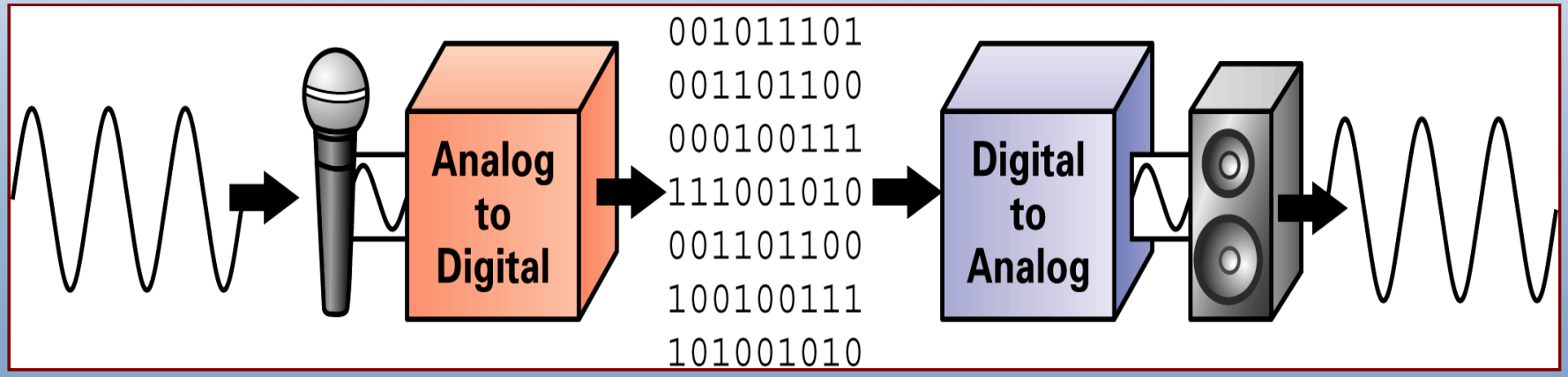
  Color can be presented as sum of intensities of three primary colors (RGB).
- *Video signal* is a sequence of images.
  It has 3D domain (time + position in 2D).

# Analog to digital conversion

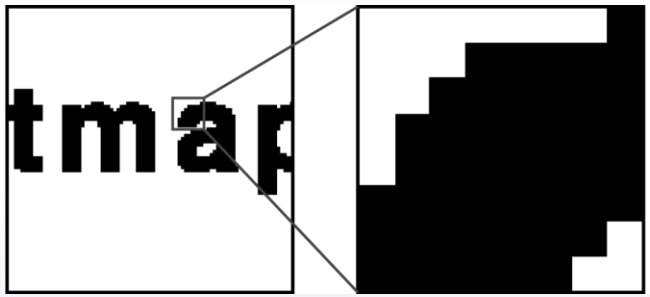
Converting Analog to Digital requires:

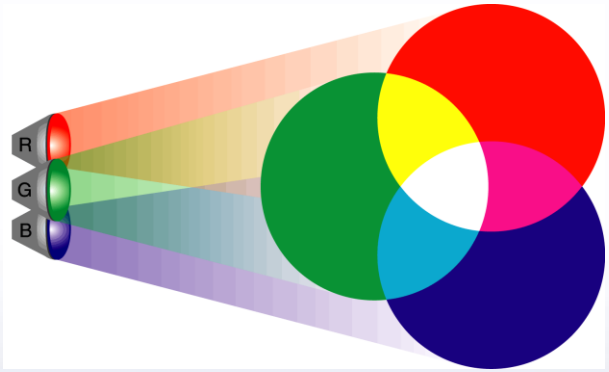
1. Sampling (signal discretization) – taking signal values at discrete time steps
2. Quantifying – representing each signal value by a number with a finite number of digits.

Digitizing sound



# Discretising pictures

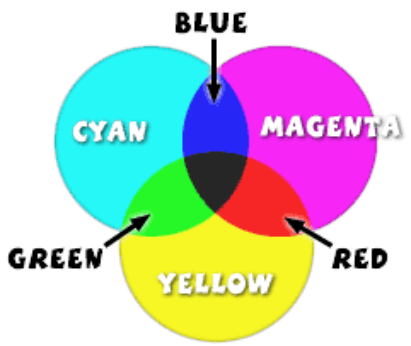


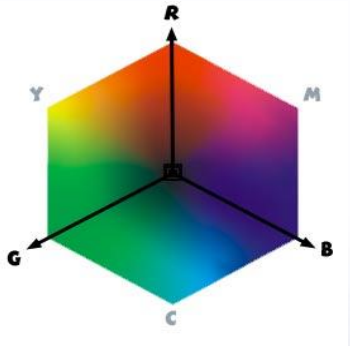




- Monitor screen is divided into a grid of small units called picture elements or **pixels**.
- For black-white image we need only digits 0 and 1 to represent image digitally.
- For color images color of each pixel on the screen is a combination of **red**, **green** and **blue** (RGB) at various intensities.

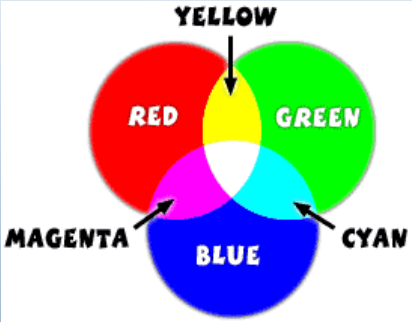
# Color models

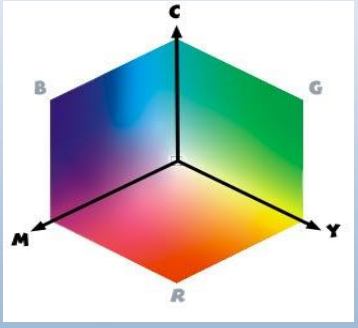
**Additive color model RGB** uses basic colors red, green, blue. All colors are produced by mixing these three basic components.

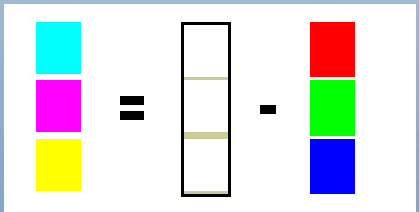



**Subtractive color model CMY** (Cyan Magenta Yellow). Subtractive the basic colors red, green, blue from white color.



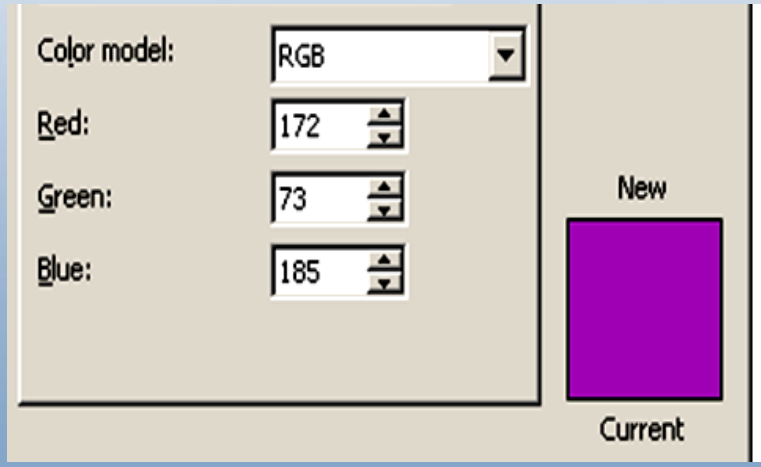



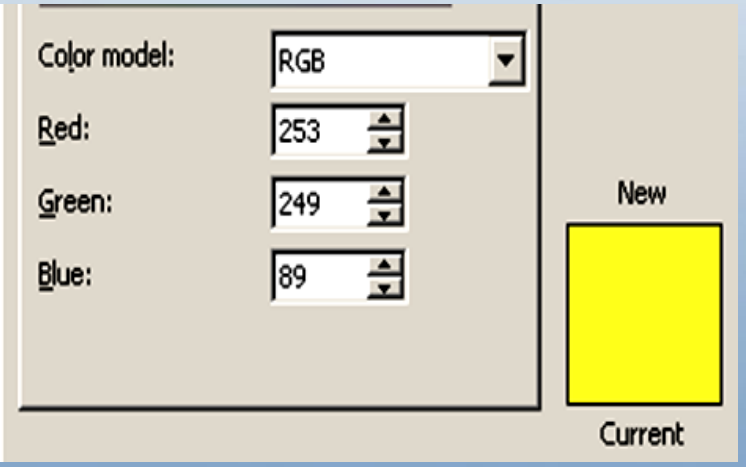



# Discrete RGB Colors

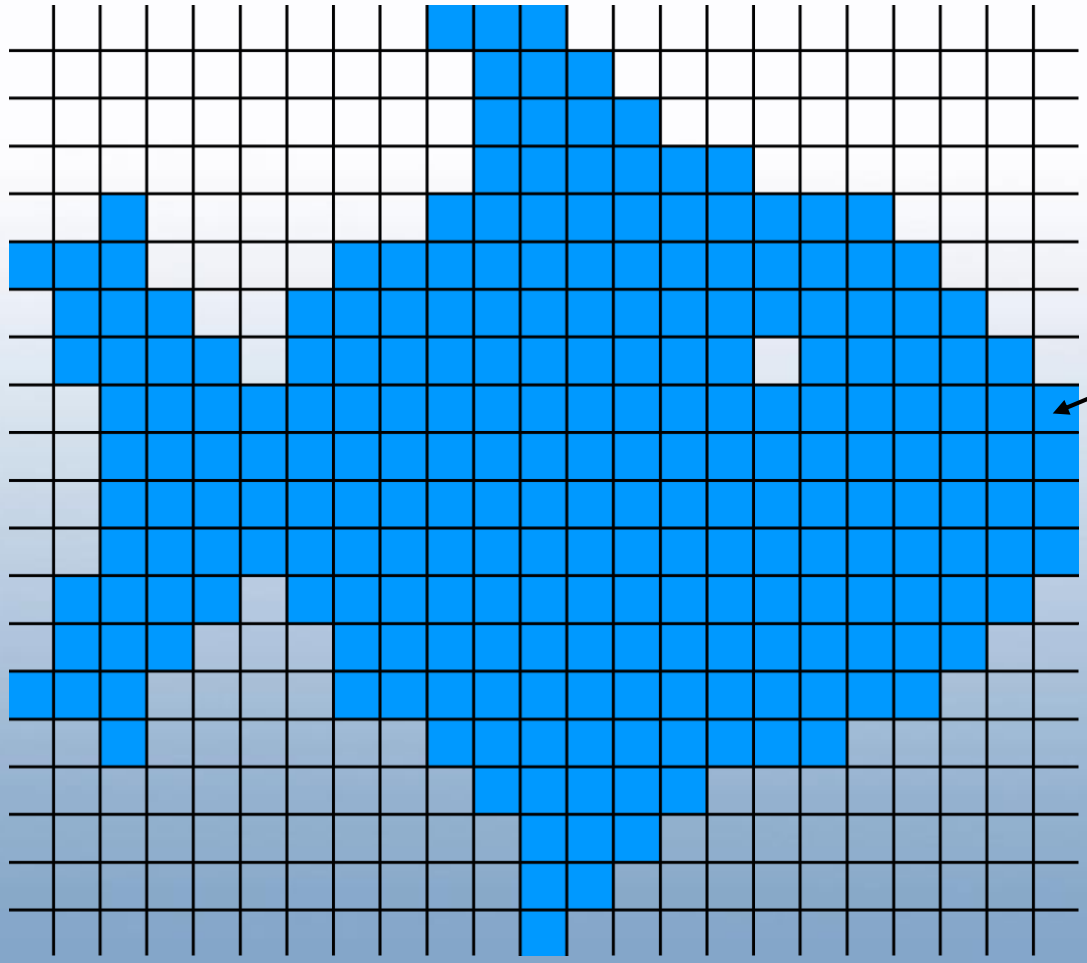
Each color intensity of red, green and blue represented as quantity from 0 through 255.

| | Red | Green | Blue |
|---|---|---|---|
| Purple: | 172 | 73 | 185 |
| Gold: | 253 | 249 | 88 |



# Image as discrete structure

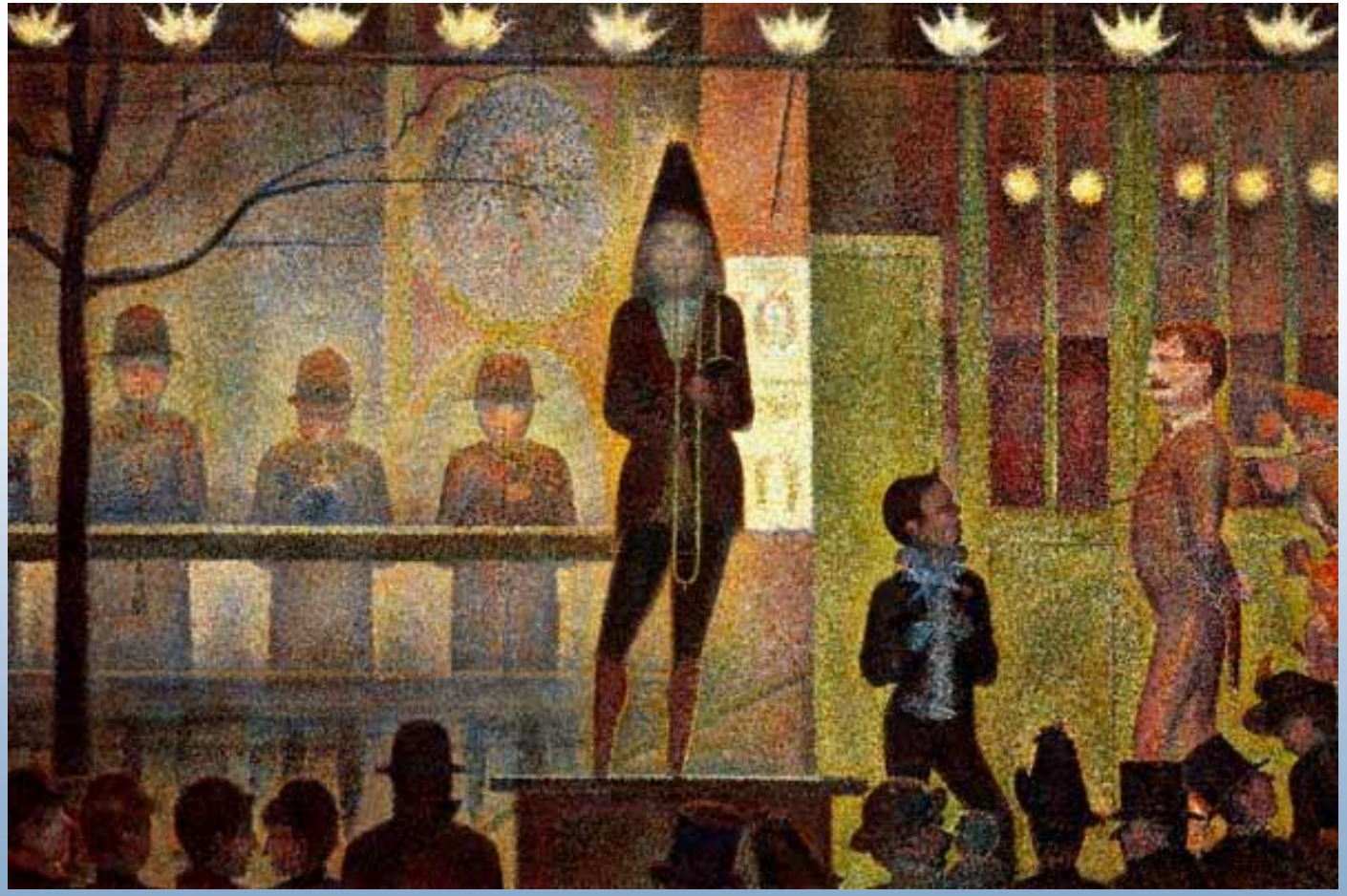


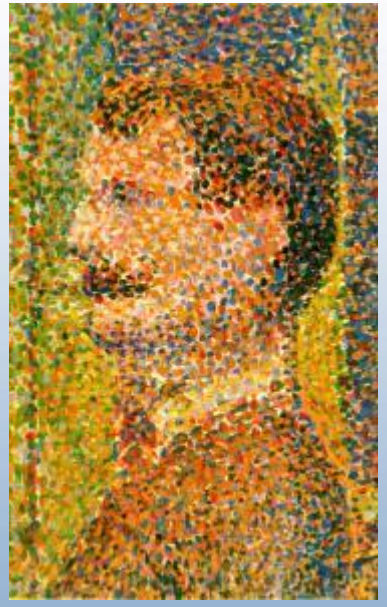
(23,165,211)

# Renaissance


Abraham Bosse, ‘Artist painting a portrait over a grid for accurate proportion’, Paris 1737

# Pointilism





Georges Seurat, 'La Parade de Cirque' (1889)

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# Notion of Structure

- "Structures" - objects built up from simpler objects according to a definite pattern.
- The structure of a thing is how the parts of it relate to each other, how it is "put together".
- Both reality and language have structure. One of the goals of science is to create and use language the structure of which accurately parallels the structure of reality.

# Discrete mathematics

Discrete Mathematics is the study of discrete mathematical objects and structures.

What are “discrete structures” ?

“Discrete” (≠ “discreet”!) - composed of distinct, seperable parts (opposite of *continuous*.)

*discrete:continuous = digital:analog*

# Discrete mathematics

• **Discrete mathematics** is the study of mathematical structures that are fundamentally discrete in the sense of not supporting or requiring the notion of continuity.

• Discrete mathematics is a theoretical foundation of digital computing.

• In contrast, traditional **mathematical analysis** studies properties of mathematical objects on the basis of their continuity.

# Discrete mathematics

Main fields of discrete mathematics:

- Logic
- Set theory
- Numbers theory
- Combinatorics
- Graph theory
- Automata
- Models of computation
- Algorithmics

# Contents

- Digital vs analog = discrete vs continuous
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- Some discrete structures & fields of discrete mathematics
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# Some discrete structures

- Propositions
- Predicates
- Proofs
- Sets
- Numbers
- Functions
- Relations
- Algorithms
- Summations
- Permutations
- Combinations
- Graphs
- Trees
- Automata

# Propositions and Predicates

In logic, a proposition is a *statement* with some definite meaning and a truth value that is either true or false.

Examples

**Apples are a type of fruit** – true

**Dolphins are not mammals** – false

Predicate is a proposition with a variable:

$P(X) = X > 3$

Its truth value is not known without the X value:

X=5: X > 3 true

X=3: X > 3 false

# Logic

Logic is the study of the principles and criteria of valid inference - constructing new statements on the basis of existing ones.

Rules of proper inference when correctly applied to true premises, lead to true conclusions.

Predicate Logic is a formal notation for defining any mathematical theory.

Mathematical Logic studies formal features of logical inference using symbolic abstractions and provides the foundation for expressing formal proofs in all branches of mathematics.

# Sets

- A *set* is any well defined collection
  (list, group, tribe, herd, flock, pack, pod)
  of objects, in which the order and multiplicity of objects has no significance
- The objects are called the *elements* or *members* of the set.

# Set membership

Set A: birds

X is an element of A

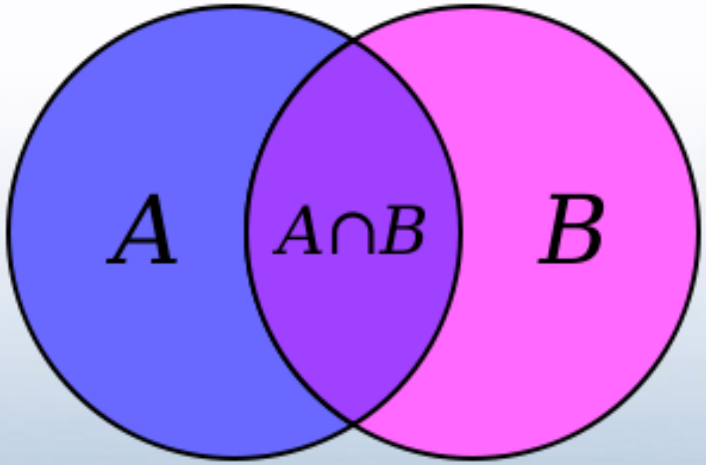
True

False

# Set theory

Set theory is the branch of discrete mathematics that concerns the study of sets, operations on sets, relations between sets and set properties.

*All* of mathematics can be defined in terms of some form of set theory.



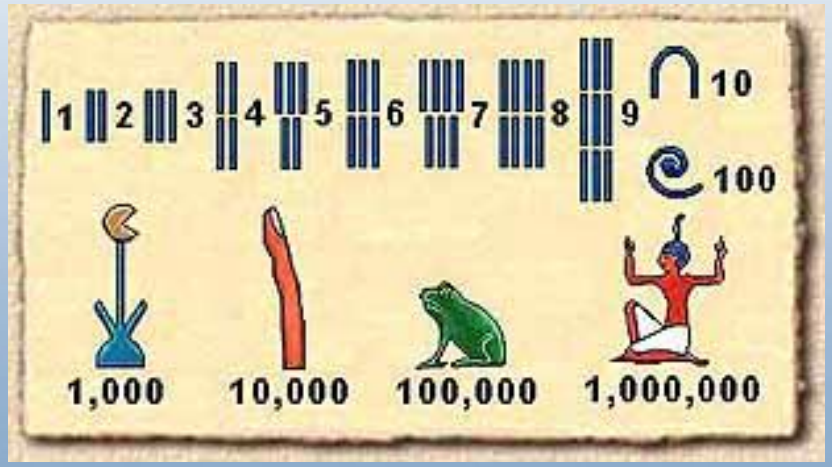

Venn diagram for intersection of two sets

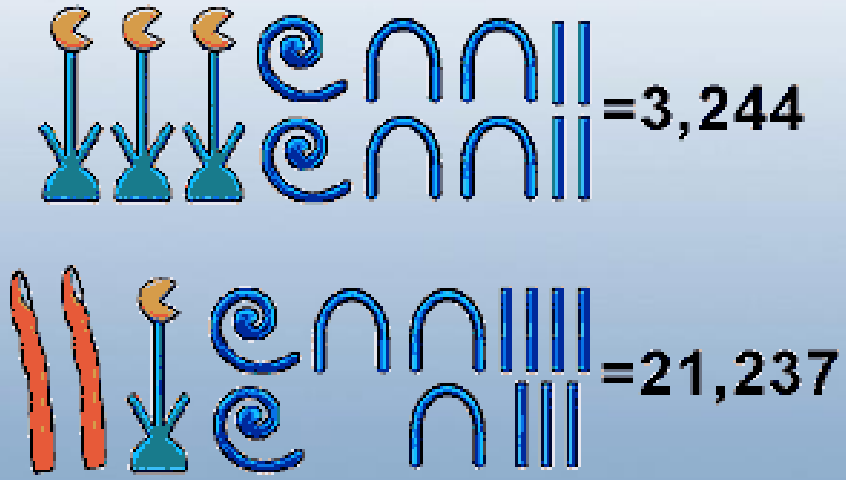
# Numbers

A *number* is an abstract idea and a set of symbols used in counting and measuring.

A *numeral system* is a writing system for expressing numbers, using symbols in a consistent manner.

## Egyptian numeral system




http://www.eyelid.co.uk/numbers.htm

# Number system

A *number system* is a set of numbers together with one or more operations (addition or multiplication).

**Number systems**

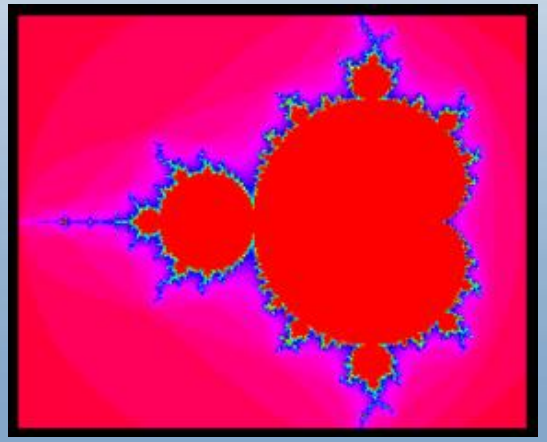
| | | |
|---|---|---|
| $\mathbb{N}$ | **Natural** | 0, 1, 2, 3, 4, ... **or** 1, 2, 3, 4, ... |
| $\mathbb{Z}$ | **Integers** | ..., −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, ... |
| - | **Positive integers** | 1, 2, 3, 4, 5, ... |
| $\mathbb{Q}$ | **Rational** | $a/b$ where $a$ and $b$ are integers and $b$ is not zero |
| $\mathbb{R}$ | **Real** | The limit of a convergent sequence of rational numbers |
| $\mathbb{C}$ | **Complex** | $a + bi$ where $a$ and $b$ are real numbers and $i$ is the square root of −1 |

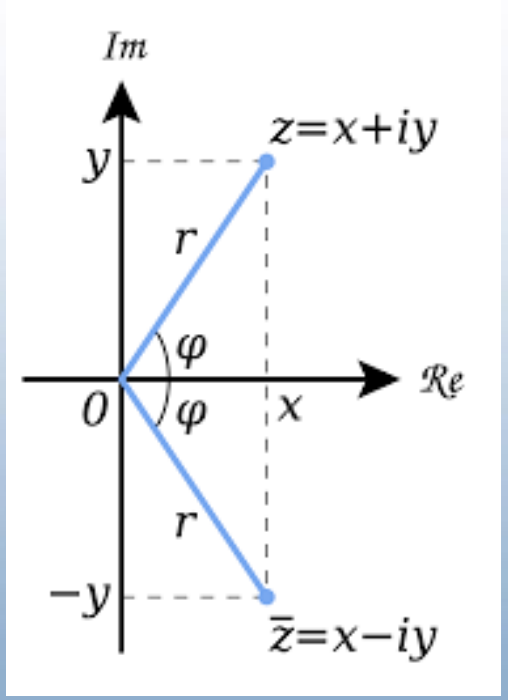
# Complex numbers and fractals

Imaginary unit: $i^2 = -1, i = \sqrt{-1}$

Complex number: $z = x + iy$

Fractal: $z_{n+1} = f(z_n)$





# Number theory

Number theory is a branch of discrete mathematics concerned with the properties of numbers in general, and integers in particular.

Branches:

Elementary number theory (arithmetic)

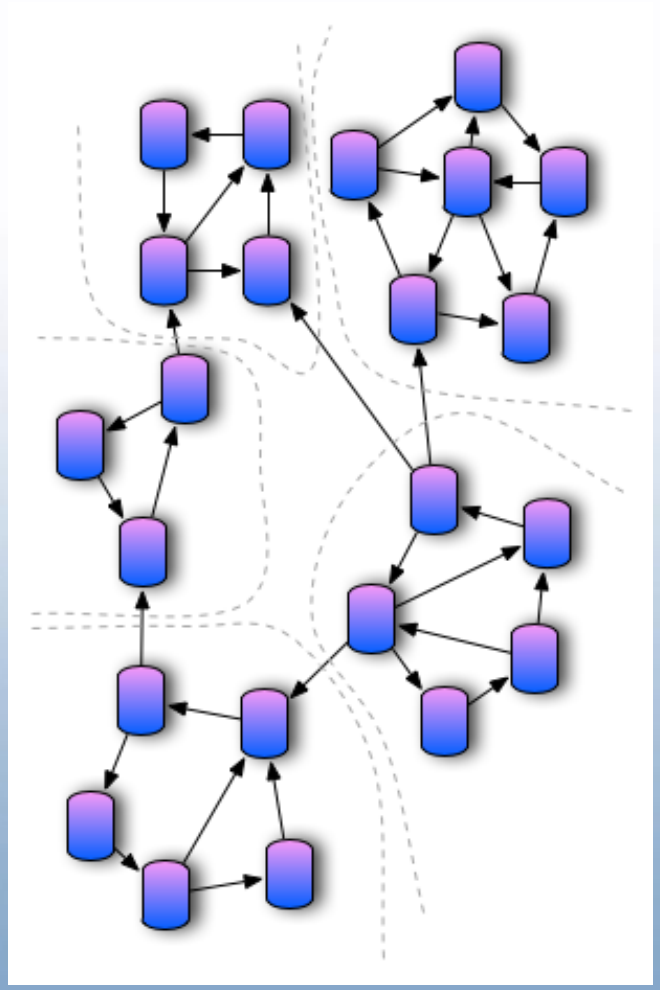
Complex numbers theory

…

# Graph theory

Graph theory studies discrete structures used to model pairwise relations between objects from a certain collection.

A *graph* is a collection of *nodes* with *edges* that connect pairs of nodes.



# Linear structures

Structure is linear if it has two properties:

P1: Each element is followed by at most one other element

P2: No two elements are followed by the same element

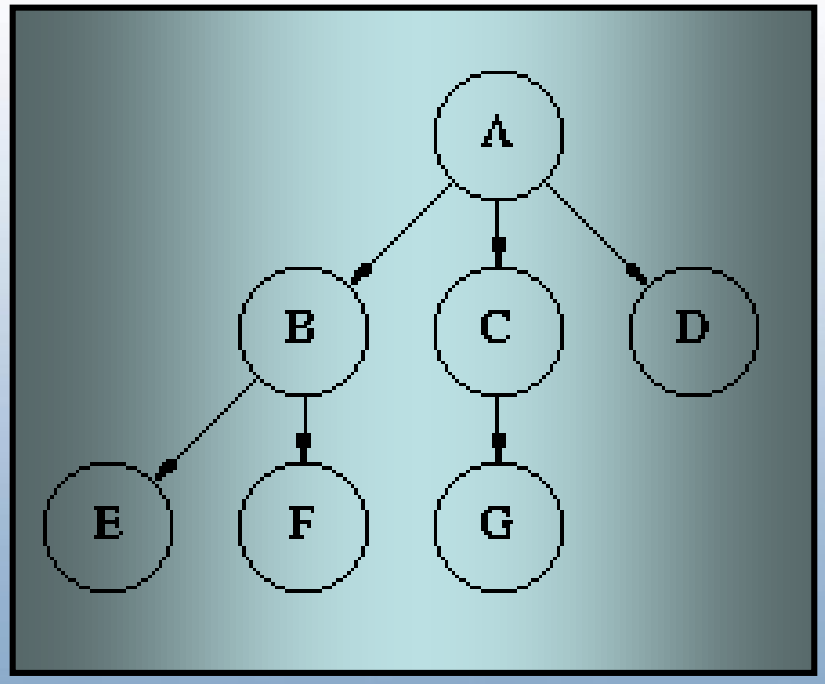
An array is a linear structure: **A→B→C→D**

Counter example 1 (violates P1): **B←A→C**

Counter example 2 (violates P2): **A→ C←B**

# Trees or Hierarchies

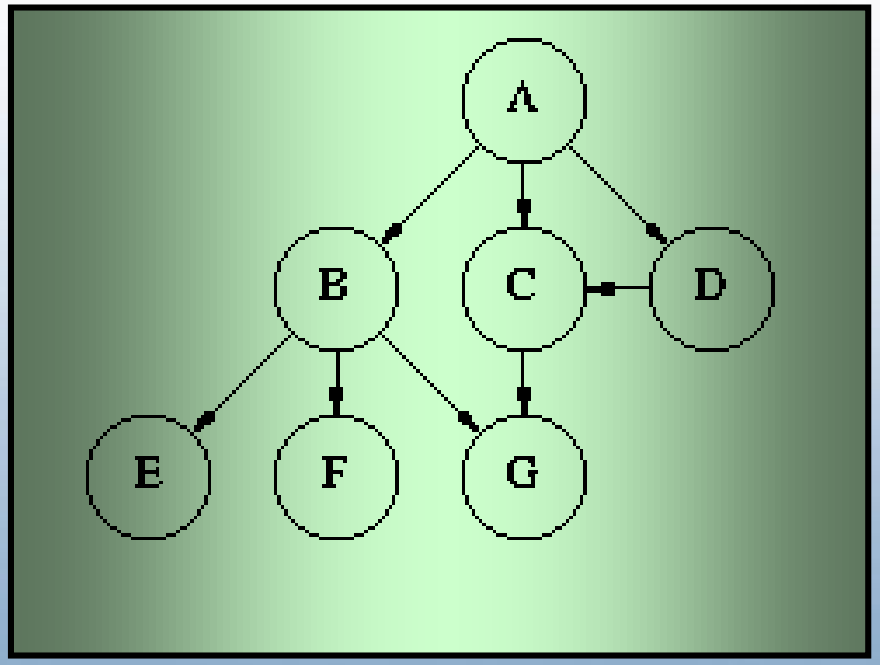
Dropping Constraint P1: tree structure or hierarchy




Counter example 1 is a tree
Counter example 2 is not a tree.

# General graphs

Dropping Constraints P1 and P2: graphs




$A \rightarrow C$, $A \rightarrow D$, $D \rightarrow C$

In a graph, there are no constraints on the relations we can define.

# Combinatorics

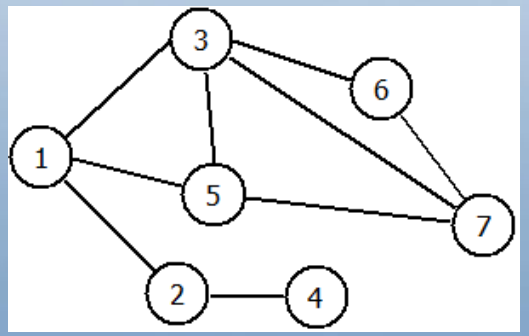
Combinatorics covers techniques of arranging objects according to the specific rules.

Examples:

- Enumeration: for the 26-letter English alphabet, how many 5-letter “words” can be constructed, if repetition of symbols is not allowed?

  Using the **permutations** formula we have 7893600 words!

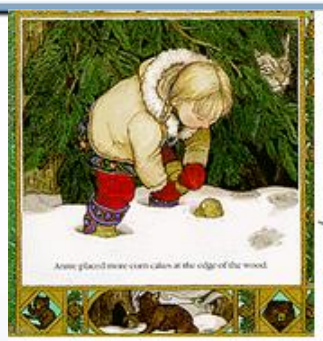
- Discrete structures: in what order a **salesman** should visit all the towns starting from 1 so that he goes through each of them at least once and returns to 1 at the end?



# Algorithm

A general algorithm is a finite set of precise instructions for solving a problem.

Algorithm = recipe



Annie's Corn Cakes

my favorite recipe ∴ by Jan Brett

warm up oven to 400° F

mix together :

1¼ c flour
¾ c Corn Meal
¼ c sugar
2 teaspoons baking powder
¼ teaspoon salt

then stir in :

1 c milk
¼ c olive oil
1 egg (beaten)

pour batter into a greased pan, and bake for 25 minutes. The pan can be a 8" or 9" baking pan. My favorite is a pan with corn cob shapes. If I use it, I don't bake them as long.

http://www.janbrett.com/annies_favorite_recipe.htm

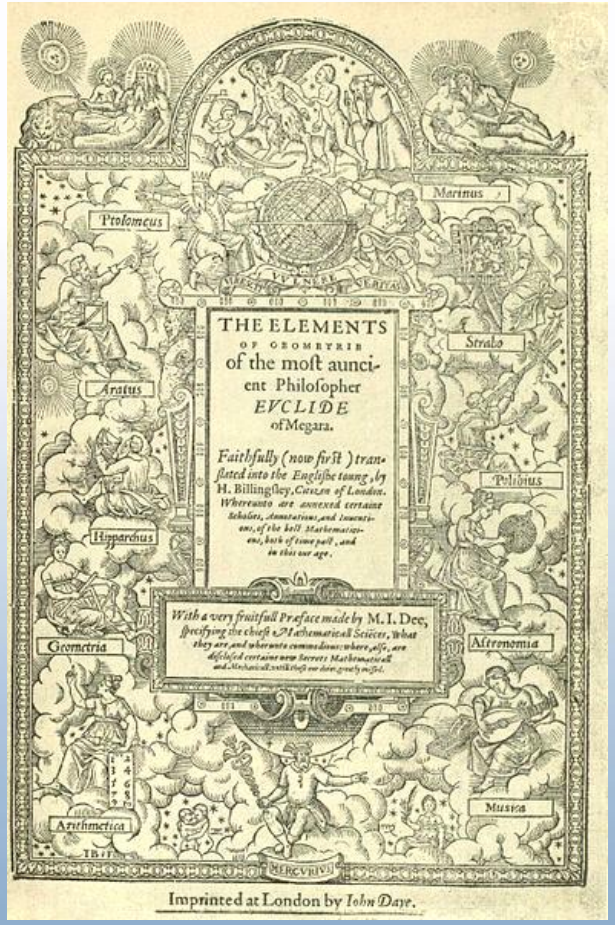
# Algorithm

- The term *algorism* was derived from the name *Al-Khwārizmī*, Persian mathematician of 9th century, who introduced the decimal positional number system.
- The word algebra comes from al-jabr, part of the title of his book *Kitab al-jabr w'al muquabala.*
- The word *algorism* was used for the rules for performing arithmetic using decimal notation


Muḥammad ibn Mūsā al-Khwārizmī

# Algorithm

- The concept of algorithm originated as means of recording procedures for solving mathematical problems such as finding common divisor of two numbers (Euclid, Elements, books VII and X, 300 BC).
- In discrete mathematics, an algorithm is a finite list of well-defined instructions for calculating a function.




First English version of Euclid's *Elements*, 1570

# Automata Theory

- Study of abstract computing devices, or “machines”, and problems they are able to solve: models of computing
- Automata – plural of “automaton”, a self-operating machine
- Automaton general behavior:
  - has internal states
  - input external “commands”
  - jumps between states depending on input commands

# Example: a Door

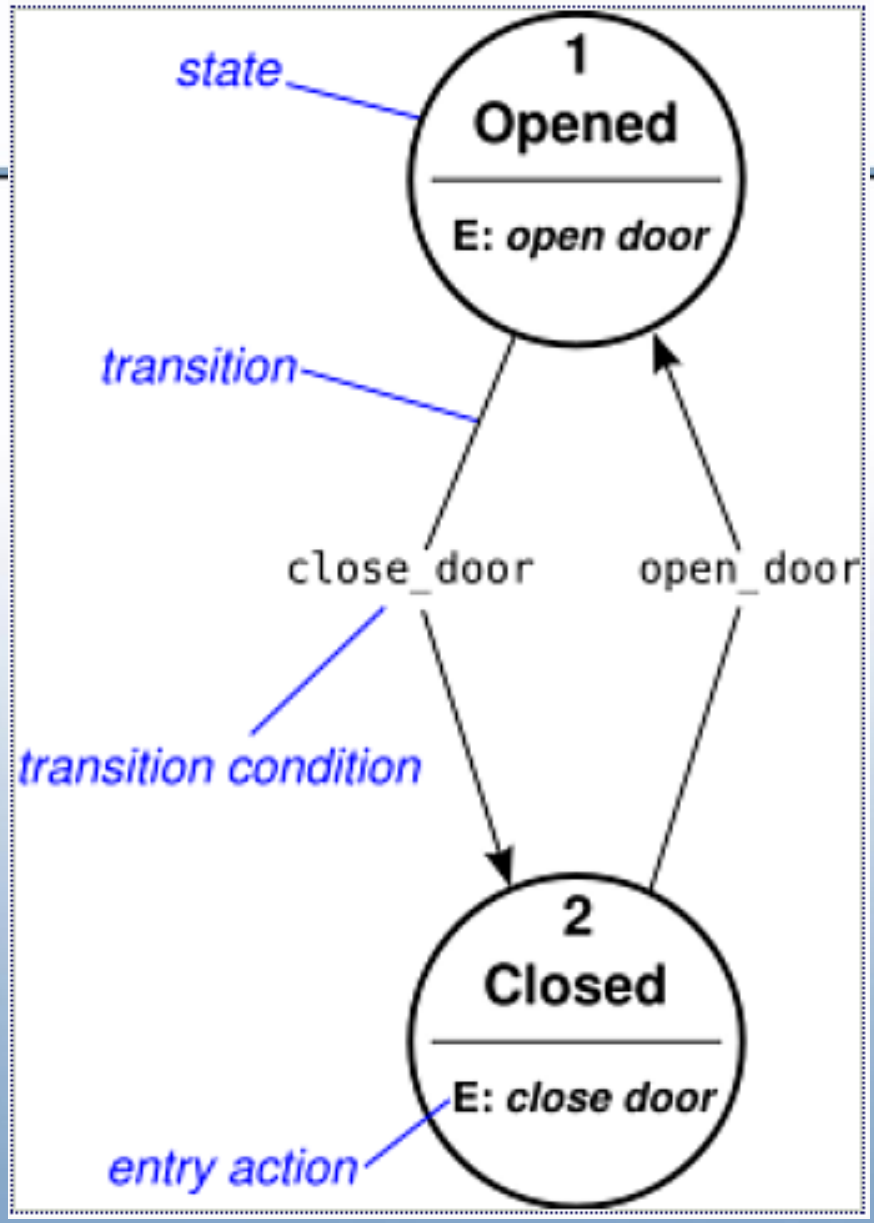
States:

Closed

Opened

French Door Refrigerator by General Electric

# Example: a Door




http://en.wikipedia.org/wiki/Finite_state_machine

## State graph

States:

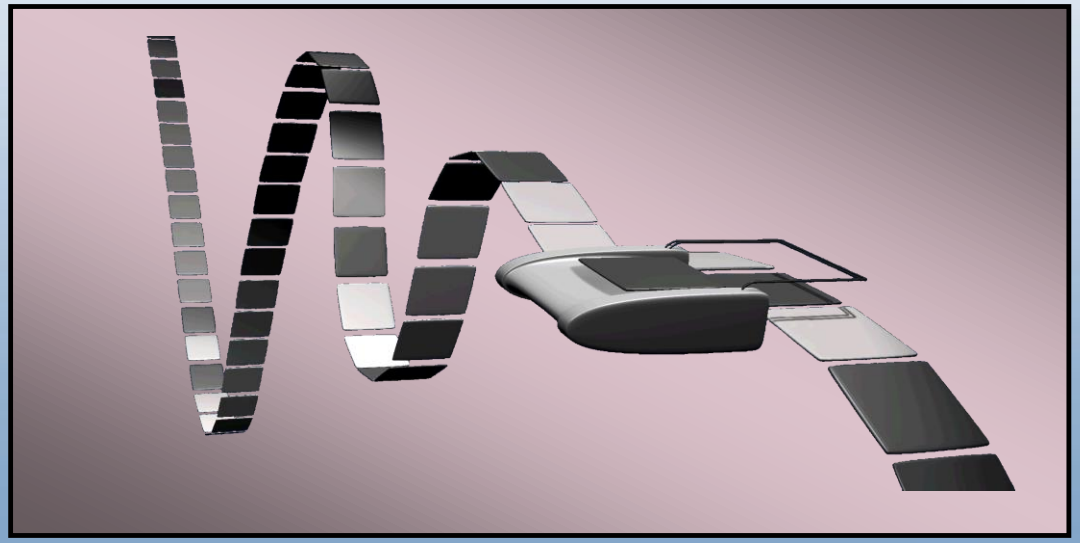
1. Opened
2. Closed

Input commands:

- Close door
- Open door

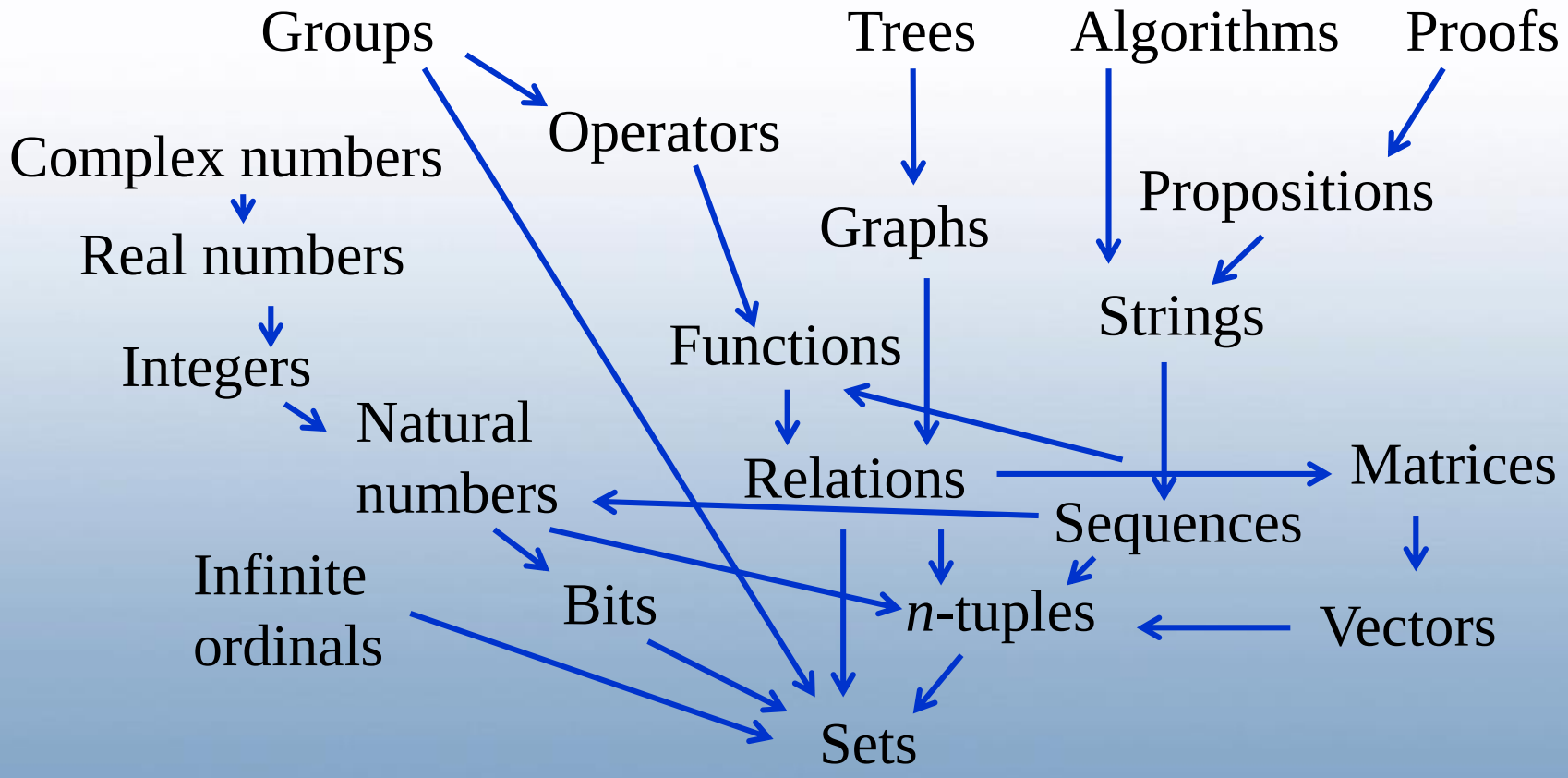
# Turing Machine

- Concept of algorithm was formalized in 1936 through Alan Turing's Logical Computing Machine and Alonzo Church's lambda calculus, which in turn formed foundation of computer science
- A Turing Machine consists of a control unit with a read/write head that can move along, read and write symbols on an infinite tape



# Relationships between discrete structures

Symbol "→" means "Can be defined in terms of"



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# Why study discrete math?

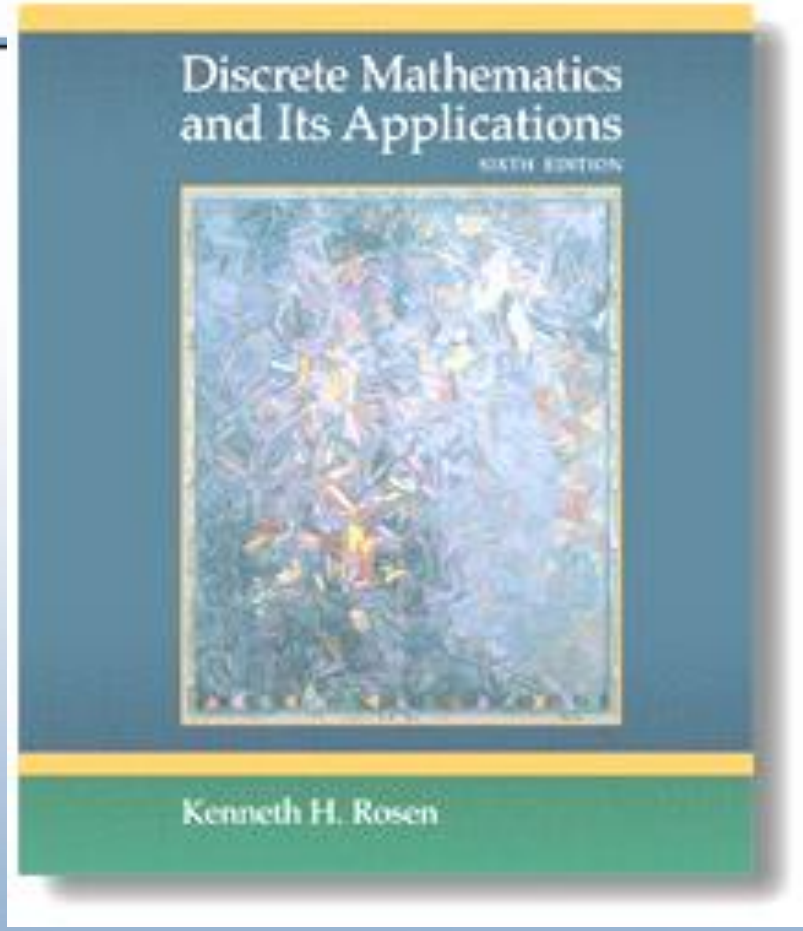
- The basis of all digital information processing is: *Discrete manipulations of discrete structures represented in memory.*
- It's the basic language and conceptual foundation of all of computer science.
- Discrete concepts are also widely used throughout math, science, engineering, economics, biology, *etc.*, …
- A generally useful tool for rational thought!

# Uses for Discrete Math in Computing

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer architecture
- Computer networks
- Software methodology and engineering
- Operating systems
- Artificial intelligence and robotics
- Database management systems
- Numeric and symbolic computations
- Cryptography
- Error correction codes
- Graphics & animation algorithms
- Game engines
- …

# References

- Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, McGraw Hill



# *Questions?*